Complex I and II Notes

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1 Complex I

1.1 Arithmetic in C

Definitions: conjugate, modulus
Main Idea: The most basic section.

We begin with the construction of the complex numbers $\mathbf{C}$. (Note that this is one of many!) Let $\mathbf{R}^2 = \{(x, y) \mid x, y \in \mathbf{R}\}$. The complex plane $\mathbf{C}$ is $\mathbf{R}^2$ with two algebraic operations, addition and multiplication, defined by

$$\begin{aligned} (x, y) + (x', y') &= (x + x', y + y'), \text{ and} \\ (x, y) \cdot (x', y') &= (xx' - yy', xy' + x'y). \end{aligned}$$

There are certain conventions; we write $(1, 0)$ as 1 and $(0, 1)$ as i. Therefore, $(x, y) = x + iy$.

To make $\mathbf{C}$ a vector space over $\mathbf{R}$, we define scalar multiplication as $\alpha(x, y) = (\alpha x, \alpha y)$ for all $\alpha \in \mathbf{R}$.

Observe that we have associativity:

$$\begin{aligned} (x + iy) + (x' + iy') &= (x + x') + i(y + y'), \text{ and} \\ (x + iy)(x' + iy') &= xx' - yy' + i(xy' + x'y). \end{aligned}$$

In fact, if we were to calculate this product *formally*, we see that

$$(x + iy)(x' + iy') = xx' + i(xy' + x'y) + i^2 yy'.$$

This implies, given the convention we are using, that i^2 should be -1.

Typically, the convention is to use the letters $z = x + iy$, $w = u + iv$, $\zeta = \xi + i\eta$, etc. We call x the real part of of z, denoted by $\operatorname{Re} z$, and y is the imaginary part of z, i.e., $y = \operatorname{Im} z$.

Definition 1.1.1. The **conjugate** of $z = x + iy$ is $\overline{z} = x - iy$.

Note that $z + \overline{z} = x + iy + x - iy = 2x$ and $z - \overline{z} = x + iy - x + iy = 2iy$, which is useful for isolating $\operatorname{Re} z$ or $\operatorname{Im} z$.

Further observe that $z \in \mathcal{C}$ is real if and only if $z = \overline{z}$, and z is purely imaginary if and only if $z = -\overline{z}$.

Now, recall that if $z = (x, y)$, $d((x, y), (0, 0)) = \sqrt{x^2 + y^2}$.

Definition 1.1.2. The **modulus** of z is denoted $|z| = \sqrt{x^2 + y^2}$.

Observe that $z\overline{z} = (x + iy)(x - iy) = x^2 + ixy - ixy - i^2y^2 = x^2 + y^2 = |z|^2$.

Lemma 1.1.3. *The following hold:*

1. $|zw| = |z||w|$,
2. $|\operatorname{Re} z| \leq |z|$, *and*
3. $|\operatorname{Im} z| \leq |z|$.

Proof. For 1., $|zw|^2 = (zw)(\overline{zw}) = zw\overline{z}\overline{w} = z\overline{z}w\overline{w} = |z|^2|w|^2$. For 2., $|\operatorname{Re} z| = |x| \leq \sqrt{x^2 + y^2} = |z|$. The proof of 3. is the same. $\square$

Next, define $0 \in \mathbf{C}$ to be $0 = 0 + i0$. Then $0z = z0 = 0$, and $-z = -x - iy$, which implies $z + (-z) = 0$. Thus $-z$ is the additive inverse of z.

Since $1 = 1 + i0$, it follows that $z = 1z = z1$.

If $z \neq 0$, then $|z| \neq 0$. Hence, if $z \neq 0$,

$$z\frac{\overline{z}}{|z|^2} = \frac{z\overline{z}}{|z|^2} = \frac{|z|^2}{|z|^2} = 1,$$

so $\dfrac{\overline{z}}{|z|^2}$ is the multiplicative inverse of z, so define $\dfrac{1}{z} = \dfrac{\overline{z}}{|z|^2}$.

In fact, in general, $\dfrac{z}{w} = z\dfrac{1}{w} = \dfrac{z\overline{w}}{|w|^2}$.

By above, we see that $\mathbf{C}$ is a field.

We can naturally embed $\mathbf{R} \subseteq \mathbf{C}$ by identifying x with $x + i0$.

1.2 The Exponential Function

Definitions:

Main Idea: Define the exponential function and give a little justification for why the definition is good. We'll show this explicitly in the future.

We define the exponential function, $\mathbf{C} \to \mathbf{C}$, $z \mapsto e^z$, as follows:

1. If $z = x$ is real, then $e^z = e^x = \displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!}$.
2. If $z = iy$ is purely imaginary, then $e^z = e^{iy} = \cos y + i \sin y$.
3. If $z = x + iy$, then $e^z = e^{x+iy} = e^x(\cos y + i \sin y)$.

While we have yet to show that this definition is a good one, we can *formally* demonstrate why this must work. Recall that $\cos y = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} y^{2k}$ and $\sin y = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} y^{2k+1}$. Further see that $i^{2k} = (i^2)^k = (-1)^k$ and that $i^{2k+1} = i^{2k} i = i(-1)^k$. Therefore, *formally*,

$$e^{iy} = \sum_{k=0}^{\infty} \frac{(iy)^k}{k!} = \sum_{k=0}^{\infty} \frac{(iy)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(iy)^{2k+1}}{(2k+1)!} = \cos y + i \sin y,$$

as claimed. The third definition should follow from the fact that $e^{a+b} = e^a e^b$. If our complex exponential function is any good at all, it should respect these properties from the real exponential.

1.3 Polar Coordinates

Definitions:

Main Idea: Polar coordinates are a very handy way to talk about complex numbers. In particular, we can separate z into its modulus and its argument.

Having defined an exponential function, we can naturally begin to talk about polar coordinates in $\mathbf{C}$.

Let $z \in \mathbf{C} \setminus \{0\}$. Then $z = |z|\frac{z}{|z|} = |z|\zeta$, where $\zeta = \frac{z}{|z|}$ has modulus 1. Now, since $e^{i\theta} = \cos\theta + i\sin\theta$, there exists θ such that $\zeta = e^{i\theta}$. Let $r = |z|$. This gives $z = re^{i\theta}$, a polar representation of z.

If $z = x + iy$, we see that $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$. Thus θ is the signed angle between the positive x-axis and the ray from 0 through z.

Note that $e^{i\theta} = \cos\theta + i\sin\theta = \cos(\theta + 2\pi k) + i\sin(\theta + 2\pi k)$ for $k \in \mathbf{Z}$. Therefore $e^{i\theta} = e^{i(\theta + 2\pi k)}$. Therefore, the polar representation of a complex number is not unique.

Let's see an application of polar coordinates:

Example 1.3.1. See that $(e^z)^n = e^{nz}$, which implies that $(re^{i\theta})^n = r^n e^{in\theta}$. Using this fact, we can find the fifth roots of 3 in $\mathbf{C}$.

We'd like $(re^{i\theta})^5 = 3 = 3e^{i2\pi k}$ for $k \in \mathbf{Z}$.

$$\begin{aligned}
\text{So } r^5e^{i5\theta} &= 3e^{i0}, & \text{so } z &= 3^{\frac{1}{5}}, \\
\text{and } r^5e^{i5\theta} &= 3e^{i2\pi}, & \text{so } z &= 3^{\frac{1}{5}}e^{i\frac{2}{5}\pi}, \\
\text{and } r^5e^{i5\theta} &= 3e^{i4\pi}, & \text{so } z &= 3^{\frac{1}{5}}e^{i\frac{4}{5}\pi}, \\
\text{and } r^5e^{i5\theta} &= 3e^{i6\pi}, & \text{so } z &= 3^{\frac{1}{5}}e^{i\frac{6}{5}\pi}, \\
\text{and } r^5e^{i5\theta} &= 3e^{i8\pi}, & \text{so } z &= 3^{\frac{1}{5}}e^{i\frac{8}{5}\pi}, \\
\text{and } r^5e^{i5\theta} &= 3e^{i10\pi}, & \text{but } z &= 3^{\frac{1}{5}}e^{i\frac{10}{5}\pi} = 3^{\frac{1}{5}}.
\end{aligned}$$

Note that the roots' arguments are evenly spaced about the circle of radius $\sqrt[5]{3}$.

A remark about computation: under multiplication, we multiply moduli and add arguments (polar coordinates make this clear).

1.4 Very Common Inequalities

Definitions:
Main Idea: It wouldn't be an analysis class without these inequalities. They hold for metric spaces other than $\mathbf{C}$, but in particular, they hold for $\mathbf{C}$.

Lemma 1.4.1 (The Triangle Inequality). *If z and w are complex numbers, then $|z+w| \leq |z| + |w|$.*

Proof. It is often handier to deal with the modulus squared. We do so here.

$$\begin{aligned}
|z+w|^2 &= (z+w)(\overline{z}+\overline{w}) \\
&= |z|^2 + z\overline{w} + \overline{z}w + |w|^2 \\
&= |z|^2 + 2\,\mathrm{Re}(z\overline{w}) + |w|^2 \\
&\leq |z|^2 + 2|z||w| + |w|^2 \\
&= (|z|+|w|)^2,
\end{aligned}$$

and the claim follows. □

Lemma 1.4.2 (The Cauchy-Schwarz Inequality). *If $z_1, ..., z_n \in \mathbf{C}$ and $w_1, ..., w_n \in \mathbf{C}$, then*

$$\left|\sum_{j=1}^{n} z_j\overline{w_j}\right|^2 \leq \left(\sum_{j=1}^{n}|z_j|^2\right)\left(\sum_{k=1}^{n}|w_k|^2\right).$$

Proof. First, if $\sum_{j=1}^{n} z_j\overline{w_j} = 0$, the result is obvious. The right hand side is a sum of nonnegative numbers, hence greater than or equal to zero. Otherwise, assume that $\sum_{j=1}^{n} z_j\overline{w_j} \neq 0$. In particular, we may assume that $\sum_{j=1}^{n}|w_j|^2 \neq 0$.

Set $\lambda = \dfrac{\sum_{j=1}^n z_j \overline{w_j}}{\sum_{j=1}^n |w_j|^2}$. Then

$$\begin{aligned} 0 &\le \sum_{j=1}^n |z_j - \lambda w_j|^2 \\ &= \sum_{j=1}^n (z_j - \lambda w_j)(\overline{z_j} - \overline{\lambda}\overline{w_j}) \\ &= \sum_{j=1}^n \left(|z_j|^2 + |\lambda|^2 |w_j|^2 - 2\operatorname{Re}\left(z_j \overline{\lambda}\overline{w_j}\right) \right) \\ &= \sum_{j=1}^n |z_j|^2 + |\lambda|^2 \sum_{j=1}^n |w_j|^2 - 2\operatorname{Re}\left(\sum_{j=1}^n z_j \overline{\lambda}\overline{w_j} \right). \end{aligned}$$

Therefore, substituting for λ,

$$\begin{aligned} 0 &\le \sum_{j=1}^n |z_j|^2 + \frac{\left|\sum_{j=1}^n z_j \overline{w_j}\right|^2}{\left(\sum_{j=1}^n |w_j|^2\right)^2} \sum_{j=1}^n |w_j|^2 - 2\operatorname{Re}\left(\overline{\left(\frac{\sum_{j=1}^n z_j \overline{w_j}}{\sum_{j=1}^n |w_j|^2} \right)} \sum_{j=1}^n z_j \overline{w_j} \right) \\ &= \sum_{j=1}^n |z_j|^2 - \frac{\left|\sum_{j=1}^n z_j \overline{w_j}\right|^2}{\sum_{j=1}^n |w_j|^2}. \end{aligned}$$

Rearranging then proves the inequality. □

One may rightfully ask, why such a λ? We remark that the λ in the proof of **Theorem 1.4.2** is a minimizer of $\sum_{j=1}^n |z_j - \lambda w_j|$.

Example 1.4.3. If $\sum_{j=1}^n |z_j|^2 < \infty$, then $\sum_{j=1}^\infty \frac{|z_j|}{j}$ converges too.

To see this, observe that by **Cauchy-Schwarz 1.4.2**,

$$\left(\sum_{j=1}^n \frac{|z_j|}{j} \right)^2 \le \left(\sum_{j=1}^n |z_j|^2 \right) \left(\sum_{j=1}^n \frac{1}{j^2} \right).$$

As $n \to \infty$, the two sequences on the right hand side converge, so the left hand side converges as well as $n \to \infty$, because it is monotonic and bounded.

1.5 Complex Polynomials

Definitions: continuously differentiable, complex differentiable, holomorphic, harmonic, Laplacian
Main Idea: We define holomorphic functions by being complex differentiable. We see that the z-derivative of a holomorphic function is the complex derivative, and we see that the $\overline{z}$-derivative of a holomorphic function is 0. Also, on open disks and rectangles, holomorphic functions have holomorphic antiderivatives.

We also discuss harmonic functions, though we'll see a lot more in the second semester, once we have a good theory of holomorphic functions.

Let's consider complex valued polynomials of a complex variable. For instance, $p(z) = iz^2 + z + 2 + i$. Since $z = x + iy$, we can express p in terms of x and y:

$$p(x+iy) = i(x+iy)^2 + x + iy + 2 + i = i(x^2 - y^2) - 2xy + y + iy + 2 + i.$$

However, the converse is not true. There are polynomials that cannot be expressed solely in terms of z. For instance, $p(x,y) = x$. To see this, p is first order, so it would have to be of the form $az+b$. But, $az+b = ax+aiy+b$, so $ax+aiy+b = x$ implies $a=1$ if we equate the x coefficients, and implies $a=0$ if we equate the y coefficients. This is a contradiction, so $p(x,y)$ cannot be written in terms of z alone.

In fact, we already know $x = \mathrm{Re}(z) = \frac{z+\overline{z}}{2}$.

Note that sometimes, we can indeed write a function in x and y as a function of z: $p(x,y) = x^2 - y^2 + 2ixy = z^2$.

Complex analysis focuses on the study of these functions, those that can be written in terms of z only.

Definition 1.5.1. Let $U \subseteq \mathbf{R}^2$ and let $f : U \to \mathbf{R}$. The function f is called C^1, or **continuously differentiable**, on U if $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ both exist and are continuous on U. We write $f \in C^1(U)$. The function f is called C^k on U and written $f \in C^k(U)$ if all partial derivatives up to and including order k are continuous on U.

We remark that if given $f : U \to \mathbf{C}$, $f = u + iv$ where u and v are real-valued, then if $u, v \in C^k(U)$, f is C^k. This is as if $w \in \mathbf{C}$, then $\mathrm{Re}\, w = \frac{w+\overline{w}}{2}$ and $\mathrm{Im}\, w = \frac{w-\overline{w}}{2i}$. Both are continuous functions. Let U be a connected open subset of $\mathbf{C}$, and suppose $f : U \to \mathbf{C}$, then we can express $f(z) = u(z) + iv(z)$, where $u(z) = \mathrm{Re}(f(z))$ and $v(z) = \mathrm{Im}(f(z))$. Hence, u and v are real-valued on U, as claimed.

Definition 1.5.2. Suppose $z_0 \in U$. We say that f is **(complex) differentiable** at z_0 if

$$\lim_{z \to z_0} \frac{f(z_0) - f(z)}{z_0 - z}$$

exists in $\mathbf{C}$. In this case, we let $f'(z_0)$ designate this limit.

Definition 1.5.3. If $f'(z_0)$ exists for all $z_0 \in U$, then we say that f is **holomorphic** in U.

Complex analysis in one variable is the study of holomorphic functions! We will flesh out so many different characterizations of holomorphic functions as the notes continue.

Theorem 1.5.4 (**The Cauchy-Riemann Equations**). *If $f = u + iv$ is holomorphic in U, then the Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

hold in U.

Proof. Let $z_0 \in U$ and assume that $f'(z_0)$ exists. Then

$$\lim_{x \to 0} \frac{f(z_0 + x) - f(z_0)}{x} = f'(z_0) = \lim_{y \to 0} \frac{f(z_0 + iy) - f(z_0)}{iy},$$

as these are just limits in the purely real and purely imaginary directions. See that

$$\begin{aligned}
f'(z_0) &= \lim_{x \to 0} \frac{f(z_0 + x) - f(z_0)}{x} \\
&= \lim_{x \to 0} \frac{u(z_0 + x) + iv(z_0 + x) - (u(z_0) + iv(z_0))}{x} \\
&= \lim_{x \to 0} \left(\frac{u(z_0 + x) - u(z_0)}{x} + i \frac{v(z_0 + x) - v(z_0)}{x} \right).
\end{aligned}$$

This tells us that $\frac{\partial u}{\partial x}(z_0)$ and $\frac{\partial v}{\partial x}(z_0)$ both exist, and

$$\frac{\partial u}{\partial x}(z_0) + i \frac{\partial v}{\partial x}(z_0) = f'(z_0).$$

Also, we may write

$$\begin{aligned}
f'(z_0) &= \lim_{y \to 0} \frac{f(z_0 + iy) - f(z_0)}{iy} \\
&= -i \lim_{y \to 0} \frac{f(z_0 + iy) - f(z_0)}{y} \\
&= -i \left(\frac{\partial u}{\partial y}(z_0) + i \frac{\partial v}{\partial y}(z_0) \right) \\
&= \frac{\partial v}{\partial y}(z_0) - i \frac{\partial u}{\partial y}(z_0).
\end{aligned}$$

So we have that the x and y partials of u and v exist and for all $z_0 \in U$,

$$\frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0), \qquad \text{and} \qquad \frac{\partial u}{\partial y}(z_0) = -\frac{\partial v}{\partial x}(z_0),$$

as desired. □

We also have two differential operators:

1. $\dfrac{\partial}{\partial z} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - i \dfrac{\partial}{\partial y} \right)$, and
2. $\dfrac{\partial}{\partial \overline{z}} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + i \dfrac{\partial}{\partial y} \right)$.

Observe the following:

Suppose $f : U \to \mathbf{C}$ and f is differentiable at $z_0 \in U$. We may write $f = u + iv$. Then,

$$\begin{aligned}
\frac{\partial f}{\partial \overline{z}}(z_0) &= \frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0) + i \frac{\partial f}{\partial y}(z_0) \right) \\
&= \frac{1}{2}\left[\left(\frac{\partial y}{\partial x}(z_0) + i \frac{\partial v}{\partial x}(z_0) \right) + i \left(\frac{\partial u}{\partial y}(z_0) + i \frac{\partial v}{\partial y}(z_0) \right) \right] \\
&= \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}(z_0) - \frac{\partial v}{\partial y}(z_0) \right) + i \left(\frac{\partial u}{\partial y}(z_0) + \frac{\partial v}{\partial x}(z_0) \right) \right] \\
&= 0,
\end{aligned}$$

by **Cauchy-Riemann 1.5.4**.

Not only that, see that

$$\begin{aligned}
\frac{\partial f}{\partial z}(z_0) &= \frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0) - i \frac{\partial f}{\partial y}(z_0) \right) \\
&= \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}(z_0) + \frac{\partial v}{\partial y}(z_0) \right) + i \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}(z_0) \right) \right] \\
&= \frac{1}{2}\left(2 \frac{\partial u}{\partial x}(z_0) + i 2 \frac{\partial v}{\partial x}(z_0) \right) \\
&= f'(z_0).
\end{aligned}$$

Thus, $\frac{\partial}{\partial \overline{z}}$ and $\frac{\partial}{\partial z}$ are indeed operators, which indicates (among other things) linearity. So if a and b are scalars, and f and g are complex-valued functions on U, then

$$\begin{aligned}
\frac{\partial(af + bg)}{\partial \overline{z}} &= a \frac{\partial f}{\partial \overline{z}} + b \frac{\partial g}{\partial \overline{z}}, \text{ and} \\
\frac{\partial(af + bg)}{\partial z} &= a \frac{\partial f}{\partial z} + b \frac{\partial g}{\partial z}.
\end{aligned}$$

Furthermore, these operators obey the product rule; that is,

$$\frac{\partial(fg)}{\partial\overline{z}} = \frac{\partial f}{\partial\overline{z}}g + f\frac{\partial g}{\partial\overline{z}}, \text{ and}$$
$$\frac{\partial(fg)}{\partial z} = \frac{\partial f}{\partial z}g + f\frac{\partial g}{\partial z}.$$

Even moreso, see that

$$\frac{\partial\overline{z}}{\partial\overline{z}} = 1,\ \frac{\partial z}{\partial\overline{z}} = 0,\ \frac{\partial\overline{z}}{\partial z} = 0,\ \text{and } \frac{\partial z}{\partial z} = 1.$$

Lemma 1.5.5. *Let j and k be positive integers. Then, from the above, we have*

$$\left(\frac{\partial^\ell}{\partial z^\ell}\right)\left(\frac{\partial^m}{\partial\overline{z}^m}\right)\left(z^j\overline{z}^k\right) = j(j-1)\cdots(j-\ell+1)k(k-1)\cdots(k-m+1)z^{j-\ell}\overline{z}^{k-m},$$

where $1 \leq \ell \leq j$ and $1 \leq m \leq k$.

Proof. We prove by induction. $\frac{\partial}{\partial z}z = 1$, and induction using the product rule shows that $\frac{\partial}{\partial z}z^j = jz^{j-1}$. And since $\frac{\partial}{\partial z}\overline{z} = 0$ and $\frac{\partial}{\partial\overline{z}}z = 0$, we have

$$\frac{\partial^\ell}{\partial z^\ell}\frac{\partial^m}{\partial\overline{z}^m}\left(z^j\overline{z}^k\right) = \left(\frac{\partial^\ell z^j}{\partial z^\ell}\right)\left(\frac{\partial^m\overline{z}^k}{\partial\overline{z}^m}\right),$$

and the claim follows. □

We remark (rather obviously) that $\dfrac{\partial^{j+k}}{\partial z^j\partial\overline{z}^k}z^j\overline{z}^k = j!k!$.

Lemma 1.5.6. *Let $p(z,\overline{z}) = \sum a_{\ell m}z^\ell\overline{z}^m$ be a polynomial. Then p contains no term with $m > 0$ (i.e., no $\overline{z}$ terms) if and only if $\frac{\partial p}{\partial\overline{z}} = 0$.*

Proof. If $a_{\ell m} = 0$ whenever $m > 0$, then $p(z) = \sum a_{\ell 0}z^\ell$, and by linearity and the product rule,

$$\frac{\partial p}{\partial\overline{z}} = \sum a_{\ell 0}\ell z^{\ell-1}\frac{\partial z}{\partial\overline{z}} = 0.$$

Conversely, if $\frac{\partial p}{\partial\overline{z}} = 0$, then $\frac{\partial^{\ell+m}p}{\partial z^\ell\overline{z}^m} = 0$ also. If $m \geq 1$, by **Lemma 1.5.5**,

$$\frac{\partial^{\ell+m}p(0)}{\partial z^\ell\overline{z}^m} = a_{\ell m}\ell!m!.$$

Thus $a_{\ell m} = 0$ for $m \geq 1$. □

Proposition 1.5.7. *If $p(z,\overline{z}) = \sum a_{\ell m}z^\ell\overline{z}^m$ and $q(z,\overline{z}) = \sum b_{\ell m}z^\ell\overline{z}^m$ are polynomials and $p(z,\overline{z}) = q(z,\overline{z})$ for all z and $\overline{z}$, then $a_{\ell m} = b_{\ell m}$ for all ℓ and m.*

Example 1.5.8. Recall **Definition 1.5.2**, f is differentiable at z if $f'(z) = \lim_{h\to 0}\dfrac{f(z+h)-f(z)}{h}$ exists. Observe the following; let $g(z,\overline{z}) = \overline{z}$. Then

$$\frac{g(z+h,\overline{z+h}) - g(z,\overline{z})}{h} = \frac{\overline{z}+\overline{h}-\overline{z}}{h} = \frac{\overline{h}}{h}.$$

If we write $h = re^{i\theta}$, then $\dfrac{\overline{h}}{h} = \dfrac{re^{-i\theta}}{re^{i\theta}} = e^{-2i\theta}$. From this we can conclude that no limit as $r \to 0$ exists! The limit will differ depending on the ray of approach, as it depends solely on θ.

Lemma 1.5.9. *If $f : U \to \mathbf{C}$ is C^1 and holomorphic, then*

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

on U.

Proof. This follows from the fact that $\dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial z} + \dfrac{\partial}{\partial \overline{z}}$ and $-i\dfrac{\partial}{\partial y} = \dfrac{\partial}{\partial z} - \dfrac{\partial}{\partial \overline{z}}$, and $\dfrac{\partial f}{\partial \overline{z}} = 0$ for f holomorphic. $\square$

Now, recall the **Cauchy-Riemann 1.5.4** equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

It surely then follows that

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \qquad \text{and} \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}.$$

If we now further assume that $f = u + iv$ is C^2 and holomorphic[1], then, as f is C^2, v is as well, so

$$\frac{\partial^2 v}{\partial y \partial x} = \frac{\partial^2 v}{\partial x \partial y},$$

so we can conclude

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

This is remarkable enough to be defined:

Definition 1.5.10. If $U \subseteq \mathbf{C}$ is open and $u \in C^2(U)$, then u is called **harmonic** if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Definition 1.5.11. The operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is called the **Laplace operator** or **Laplacian**[2].

Lemma 1.5.12.

$$\Delta = 4\frac{\partial^2}{\partial z \partial \overline{z}}.$$

Proof. This is a simple computation:

$$\begin{aligned}
\frac{\partial^2 u}{\partial z \partial \overline{z}} &= \frac{1}{4}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right) \\
&= \frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2} - i\frac{\partial^2 u}{\partial x \partial y} + i\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial^2 u}{\partial y^2}\right) \\
&= \frac{1}{4}\Delta u,
\end{aligned}$$

as desired. $\square$

[1] The **Cauchy Integral Formula 1.9.3** will show us that if f is holomorphic, then not only is it C^2, it is C^∞!

[2] The Laplacian is a prototype elliptic operator; see PDEs for more.

We remark that we have seen that if $f = u + iv$ is C^2 and holomorphic, then u and v are harmonic, as the $\overline{z}$-derivative of a holomorphic function is 0.

One may naturally ask: if u is a real-valued, harmonic function, then does there exist v, real-valued and harmonic, such that $u + iv$ is holomorphic? The answer is *sometimes*. We give a partial answer now, and defer the full explanation to the second semester, when we delve deeper into harmonic functions[3].

Theorem 1.5.13. *Let R be the rectangle $R = \{(x, y) \in \mathbf{R}^2 \mid |x - a| < \delta, |y - b| < \varepsilon\}$. If $f, g \in C^1(\mathbf{R})$ and $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ on R, then there exists $h \in C^2(R)$ such that $\frac{\partial h}{\partial x} = f$ and $\frac{\partial h}{\partial y} = g$. If f and g are real-valued, then so is h.*

Proof. For $(x, y) \in R$, set

$$h(x, y) = \int_a^x f(t, b)dt + \int_b^y g(x, s)ds.$$

By the Fundamental Theorem of Calculus, $\frac{\partial h}{\partial y}(x, y) = g(x, y)$. To compute $\frac{\partial h}{\partial x}$, we use the Fundamental Theorem of Calculus and the fact that g is C^1 to differentiate inside the integral.

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\int_a^x f(t, b)dt + \int_b^y g(x, s)ds\right) &= f(x, b) + \int_b^y \frac{\partial g}{\partial x}(x, s)ds \\
&= f(x, b) + \int_b^y \frac{\partial f}{\partial s}(x, s)ds \\
&= f(x, b) + f(x, y) - f(x, b) \\
&= f(x, y).
\end{aligned}$$

Since $\frac{\partial h}{\partial y} = g \in C^1$ and $\frac{\partial h}{\partial x} = f \in C^1$, it follows that $h \in C^2(R)$. Also, the construction of h clearly guarantees a real-valued function if both f and g are real-valued. $\square$

Corollary 1.5.14. *If R is an open rectangle (or open disk) and if u is a real-valued harmonic function, then there exists a holomorphic function F on R such that* $\operatorname{Re} F = u$.

Proof. Set $f = -\frac{\partial u}{\partial y}$ and $g = \frac{\partial u}{\partial x}$. Since u is harmonic,

$$\frac{\partial f}{\partial y} = -\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial g}{\partial x}$$

Since f and g are C^1 on R, there exists a real-valued $v \in C^2(R)$ satisfying

$$\frac{\partial v}{\partial x} = f = -\frac{\partial u}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial y} = g = \frac{\partial u}{\partial x},$$

by **Theorem 1.5.13**. Then u and v satisfy the **Cauchy-Riemann 1.5.4** equations, so $F = u + iv$ is holomorphic on R. $\square$

This concludes harmonic functions for the time being; we will again see more in semester two.

Now, we turn to an interesting question: are holomorphic functions the derivatives of holomorphic functions?

Theorem 1.5.15. *If $U \subseteq \mathbf{C}$ is either an open disk or open rectangle, and F is holomorphic on U, then there exists a holomorphic function H on U such that $\frac{\partial H}{\partial z} = F$ on U.*

Proof. Write $F(z) = u(z) + iv(z)$, and set $f = u$ and $g = -v$. Then by the **Cauchy-Riemann 1.5.4** equations,

$$\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \frac{\partial g}{\partial x}.$$

[3] See, for instance, **Lemma 2.3.3**.

By **Theorem 1.5.13**, there exists a real-valued $h_1 \in C^2(U)$ such that

$$\frac{\partial h_1}{\partial x} = f = u \qquad \text{and} \qquad \frac{\partial h_1}{\partial y} = g = -v.$$

Next, set $\widetilde{f} = v$ and $\widetilde{g} = u$. Again, by **Cauchy-Riemann 1.5.4**,

$$\frac{\partial \widetilde{f}}{\partial y} = \frac{\partial v}{\partial y} = \frac{\partial y}{\partial x} = \frac{\partial \widetilde{g}}{\partial x}.$$

By **Theorem 1.5.13**, there exists a real-valued $h_2 \in C^2(U)$ such that

$$\frac{\partial h_2}{\partial x} = \widetilde{f} = v \qquad \text{and} \qquad \frac{\partial h_2}{\partial y} = \widetilde{g} = u.$$

Set $H = h_1 + ih_2$. Then $H \in C^2(U)$, and

$$\frac{\partial h_1}{\partial x} = u = \frac{\partial h_2}{\partial y} \qquad \text{and} \qquad \frac{\partial h_1}{\partial y} = -v = -\frac{\partial h_2}{\partial x},$$

so the **Cauchy-Riemann 1.5.4** equations are satisfied, and thus H is a holomorphic.

Furthermore,

$$\begin{aligned}
\frac{\partial H}{\partial z} &= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(h_1 + ih_2) \\
&= \frac{1}{2}\left(\frac{\partial h_1}{\partial x} + \frac{\partial h_2}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial h_2}{\partial x} - \frac{\partial h_1}{\partial y}\right) \\
&= \frac{1}{2}(2u) + \frac{i}{2}(2v) = F,
\end{aligned}$$

as desired. □

1.6 Complex Line Integrals

Definitions: curve, closed (curve), simple closed (curve), continuously differentiable (function on $[a, b]$), continuous (curve), continuously differentiable (curve), line integral, contour integral
Main Idea: Using our theory of integrals in $\mathbf{R}$, we build up the idea of complex integrals.

In the proof of **Theorem 1.5.13**, we relied on integrals in the horizontal and vertical directions. This is okay for integrating $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ but is too restrictive in general. We develop stronger tools.

Definition 1.6.1. A **curve** is a continuous function $\gamma : [a, b] \to \mathbf{C}$. We sometimes refer to $\widetilde{\gamma} = \{\gamma(t) \mid a \leq t \leq b\}$ as a curve, though we generally consider the function γ to be the curve.

Definition 1.6.2. A curve γ is called **closed** if $\gamma(b) = \gamma(a)$.

Definition 1.6.3. A curve γ is called **simple closed** if $\gamma|_{[a,b)}$ is closed and one-to-one.

We often write $\gamma(t) = \gamma_1(t) + i\gamma_2(t) = (\gamma_1(t), \gamma_2(t))$; i.e., as typical, it is sometimes helpful to decompose into real and purely imaginary parts.

Definition 1.6.4. A function $\varphi : [a, b] \to \mathbf{R}$ is called **continuously differentiable**, or C^1, and written $\varphi \in C^1([a, b])$, if $\varphi \in C([a, b])$, φ' exists on (a, b), and φ' has a continuous extension to $[a, b]$.

With such a $\varphi \in C^1$, we can use the Fundamental Theorem of Calculus:

$$\begin{aligned}
\varphi(b) - \varphi(a) &= \lim_{\varepsilon \to 0^+} \varphi(b - \varepsilon) - \varphi(a + \varepsilon) \\
&= \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^{b-\varepsilon} \varphi'(t)dt \\
&= \int_a^b \varphi'(t)dt.
\end{aligned}$$

Definition 1.6.5. A curve $\gamma : [a, b] \to \mathbf{C}$ is said to be **continuous** on $[a, b]$ if both γ_1 and γ_2 are for $\gamma = \gamma_1 + i\gamma_2$.

Definition 1.6.6. A curve γ is **continuously differentiable**, or C^1, if both γ_1 and γ_2 are for $\gamma = \gamma_1 + i\gamma_2$.

In the case that $\gamma \in C^1$,

$$\gamma'(t) = \frac{d\gamma}{dt} = \frac{d\gamma_1}{dt} + i\frac{d\gamma_2}{dt}.$$

Definition 1.6.7. Let $\psi = \psi_1 + i\psi_2 \in C([a, b])$. Then define the **line integral** by

$$\int_a^b \psi(t)dt = \int_a^b \psi_1(t)dt + i\int_a^b \psi_2(t)dt.$$

From the previous two definitions, it follows that if $\gamma = \gamma_1 + i\gamma_2$ is $C^1([a, b])$, then

$$\gamma(b) - \gamma(a) = \int_a^b \gamma'(t)dt.$$

Now, recall the following result from undergraduate calculus:

Lemma 1.6.8. *Let $U \subseteq \mathbf{C}$ be open and $\gamma : [a, b] \to U$ be C^1. If $f : U \to \mathbf{R}$ and $f \in C^1(U)$, then*

$$\int_a^b \left(\frac{\partial f}{\partial x}(\gamma(t))\frac{d\gamma_1}{dt} + \frac{\partial f}{\partial y}(\gamma(t))\frac{d\gamma_2}{dt}\right) dt = \int_a^b \nabla f(\gamma(t)) \cdot \gamma'(t)dt.$$

Proof. Consider the function $f \circ \gamma : [a, b] \to \mathbf{R}$. By the Fundamental Theorem of Calculus and the chain rule,

$$\begin{aligned} f(\gamma(b)) - f(\gamma(a)) &= \int_a^b \frac{d}{dt}(f(\gamma(t)))dt \\ &= \int_a^b \left(\frac{\partial f}{\partial x}(\gamma(t)){\gamma_1}'(t) + \frac{\partial f}{\partial y}(\gamma(t)){\gamma_2}'(t)\right) dt, \end{aligned}$$

which is the line integral of ∇f along γ. The result is proven. $\square$

We now motivate the coming definition. Suppose f is holomorphic on U and $f = u + iv$. Then $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. So,

$$\begin{aligned} f(\gamma(b)) - f(\gamma(a)) &= \int_a^b \left(\frac{\partial u}{\partial x}(\gamma(t))\frac{d\gamma_1}{dt} + i\frac{\partial v}{\partial x}(\gamma(t))\frac{d\gamma_1}{dt} + \frac{\partial u}{\partial y}(\gamma(t))\frac{d\gamma_2}{dt} + i\frac{\partial v}{\partial y}(\gamma(t))\frac{d\gamma_2}{dt}\right) dt \\ &= \int_a^b \left(\frac{\partial u}{\partial x}(\gamma(t))\frac{\partial\gamma_1}{dt} - \frac{\partial v}{\partial x}(\gamma(t))\frac{d\gamma_2}{dt} + i\left(\frac{\partial v}{\partial x}(\gamma(t))\frac{d\gamma_1}{dt} + \frac{\partial u}{\partial x}(\gamma(t))\frac{d\gamma_2}{dt}\right)\right) dt \\ &= \int_a^b \frac{\partial f}{\partial x}(\gamma(t)) \cdot \frac{d\gamma}{dt}(t)dt \\ &= \int_a^b \frac{\partial f}{\partial z}(\gamma(t)) \cdot \frac{d\gamma}{dt}(t)dt. \end{aligned}$$

Thus, if f is holomorphic, then

$$f(\gamma(b)) - f(\gamma(a)) = \int_a^b \frac{\partial f}{\partial z}(\gamma(t)) \cdot \frac{d\gamma}{dt}(t)dt.$$

Definition 1.6.9. Let $U \subseteq \mathbf{C}$ be open, let $F : U \to \mathbf{C}$ be continuous on U, and let $\gamma : [a, b] \to U$ be a C^1 curve. Then the complex line integral, or **contour integral**, is

$$\oint_\gamma F(z)dz = \int_a^b F(\gamma(t))\frac{d\gamma}{dt}dt.$$

Lemma 1.6.10. *Let $U \subseteq \mathbf{C}$ be open and $\gamma : [a,b] \to U$ be a C^1 curve. If f is holomorphic on U, then*

$$f(\gamma(b)) - f(\gamma(a)) = \oint_\gamma \frac{\partial f}{\partial z}(z)dz.$$

Proof. This was what we showed in motivating the defintion of a contour integral! □

Corollary 1.6.11. *If γ is a closed C^1 curve and F is holomorphic on an open $U \subseteq \mathbf{C}$, then*

$$\oint_\gamma \frac{\partial F}{\partial z} dz = 0.$$

Proof. Obviously, as $\gamma(a) = \gamma(b)$. □

Example 1.6.12. Let $f(z) = x = \operatorname{Re} z$. Let $\gamma : [0,1] \to \mathbf{C}$ be defined by $\gamma(t) = t + it^2$. Let's compute the contour integral of f over γ.

See that $f(\gamma(t)) = f(t + it^2) = t$ and $\gamma'(t) = 1 + i2t$. Thus

$$\oint_\gamma f(z)dz = \int_0^1 f(\gamma(t))\frac{d\gamma}{dt}dt = \int_0^1 t(1+2ti)dt = \left(\frac{1}{2}t^2 + i\frac{2}{3}t^3\right)\Bigg|_0^1 = \frac{1}{2} + i\frac{2}{3}.$$

Lemma 1.6.13. *If $\varphi : [a,b] \to \mathbf{C}$ is C^0, then*

$$\left| \int_a^b \varphi(t)dt \right| \leq \int_a^b |\varphi(t)|dt.$$

Proof. Set $\alpha = \int_a^b \varphi(t)dt$. If $\alpha = 0$, then there is nothing to show. Assume $\alpha \neq 0$. Let $\eta = \dfrac{\overline{\alpha}}{|\alpha|}$. Also, let $\psi(t) = \operatorname{Re}(\eta\varphi(t))$. Then

$$\varphi(t) \leq |\eta\varphi(t)| = |\eta||\varphi(t)| = |\varphi(t)|.$$

Consequently,

$$\left| \int_a^b \varphi(t)dt \right| = |\alpha| = \frac{\alpha\overline{\alpha}}{|\alpha|} = \eta\alpha = \eta\int_a^b \varphi(t)dt.$$

Since $\left|\int_a^b \varphi(t)dt\right|$ is real-valued, so is $\eta\int_a^b \varphi(t)dt$. This means that

$$\int_a^b \eta\varphi(t)dt = \int_a^b \operatorname{Re}(\eta\varphi(t))dt = \int_a^b \psi(t)dt \leq \int_a^b |\varphi(t)|dt,$$

and the proof is shown. □

Lemma 1.6.14. *Let $U \subseteq \mathbf{C}$ be open and let $f \in C(U)$. If $\gamma : [a,b] \to U$ is a C^1 curve, then*

$$\left| \oint_\gamma f(z)dz \right| \leq \sup_{t\in[a,b]} |f(\gamma(t))| \cdot \ell(\gamma),$$

where

$$\ell(\gamma) = \int_a^b \left| \frac{d\gamma}{dt}(t) \right| dt.$$

Note that $\ell(\gamma)$ is the length of γ, since $\left|\frac{d\gamma}{dt}\right| = \sqrt{({\gamma_1}')^2 + ({\gamma_2}')^2} = |({\gamma_1}'(t), {\gamma_2}'(t))|$.

Proof. See that

$$\left|\oint_\gamma f(z)dz\right| = \left|\int_a^b f(\gamma(t))\gamma'(t)dt\right|.$$

By **Lemma 1.6.13**, we have

$$\begin{aligned}\left|\int_a^b f(\gamma(t))\gamma'(t)dt\right| &\leq \int_a^b |f(\gamma(t))| \cdot |\gamma'(t)|dt \\ &\leq \sup_{t\in[a,b]} |f(\gamma(t))| \int_a^b |\gamma'(t)|dt,\end{aligned}$$

and the lemma is proven. □

One important and useful fact that we now check is the independence of a line/contour integral on its parameterization.

Lemma 1.6.15. *Let $U \subseteq \mathbf{C}$ be open and let $f : U \to \mathbf{C}$ be continuous. Suppose $\gamma : [a, b] \to U$ is a C^1 curve and $\varphi : [c, d] \to [a, b]$ is a one-to-one, onto, increasing C^1 function with a C^1 inverse. Let $\widetilde{\gamma} = \gamma \circ \varphi$; then*

$$\oint_\gamma f dz = \oint_{\widetilde{\gamma}} f dz.$$

Proof. We calculate

$$\begin{aligned}\oint_{\widetilde{\gamma}} f dz &= \int_c^d f(\widetilde{\gamma}(t))\frac{d\widetilde{\gamma}}{dt}dt \\ &= \int_c^d f(\gamma(\varphi(t)))\frac{d}{dt}\Big(\gamma_1(\varphi(t)) + i\gamma_2(\varphi(t))\Big)dt \\ &= \int_c^d f(\gamma(\varphi(t)))\left({\gamma_1}'(\varphi(t)) + i{\gamma_2}'(\varphi(t))\right)\varphi'(t)dt.\end{aligned}$$

Now let $s = \varphi(t)$; then $ds = \varphi'(t)dt$. Therefore,

$$\oint_{\widetilde{\gamma}} f dz = \int_a^b f(\gamma(s))({\gamma_1}'(s) + i{\gamma_2}'(s))ds = \oint_\gamma f dz,$$

as desired. □

Note that contour integrals are independent of parameterization, but the direction matters.

Example 1.6.16. Let $f(z) = \frac{1}{z}$. We compute the two contour integrals along $\gamma_1(t) = e^{it}$, $0 \leq t \leq 2\pi$ and along $\gamma_2(t) = e^{-it}$, $0 \leq t \leq 2\pi$. See that ${\gamma_1}'(t) = ie^{it}$ and ${\gamma_2}'(t) = -ie^{it}$. Then

$$\begin{aligned}\oint_{\gamma_1} f dz &= \int_0^{2\pi} \frac{1}{e^{it}} ie^{it}dt = 2\pi i, \text{ but} \\ \oint_{\gamma_2} f dz &= \int_0^{2\pi} \frac{1}{e^{-it}}(-ie^{-it})dt = -2\pi i.\end{aligned}$$

1.7 Complex Differentiability

Definitions: limit, continuous
Main Idea: One good result is that for holomorpic functions, $\frac{\partial f}{\partial z} = f'$. Another is that complex differentiability is holomorphicity.

Recall that the complex derivative (**Definition 1.5.2**), $f'(z)$, is

$$\lim_{w \to z} \frac{f(w) - f(z)}{w - z},$$

provided the limit exists.

Definition 1.7.1. Let $U \subseteq \mathbf{C}$ be open, $\zeta \in U$, and $g : U \setminus \{\zeta\} \to \mathbf{C}$ a function. We write

$$\lim_{z \to \zeta} g(z) = L$$

for $L \in \mathbf{C}$ and say the **limit** of $g(z)$ as $z \to \zeta$ is L if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $z \in U$ and $0 < |z - \zeta| < \delta$, then $|g(z) - L| < \varepsilon$.

Definition 1.7.2. We say $f : U \to \mathbf{C}$ is **continuous** at ζ if

$$\lim_{z \to \zeta} f(z) = f(\zeta).$$

Theorem 1.7.3. *Let $U \subseteq \mathbf{C}$ be open and let f be holomorphic on U. Then f' exists at each point of U, and $f'(z) = \dfrac{\partial f}{\partial z}(z)$ for all $z \in U$.*

Proof. Let $z_0 \in U$. If z is near enough to z_0, then $\ell(z, z_0)$, the line segment from z to z_0, is a subset of U, as U is open.

Set $\gamma(t) = (1-t)z_0 + tz$. So $\gamma : [0,1] \to U$, and $\gamma(0) = z_0$ and $\gamma(1) = z$, and $\gamma'(t) = z - z_0$. By **Lemma 1.6.10**,

$$f(z) - f(z_0) = f(\gamma(1)) - f(\gamma(0)) = \oint_\gamma \frac{\partial f}{\partial z} dz = \int_0^1 \frac{\partial f}{\partial z}(\gamma(t)) \cdot \frac{d\gamma}{dt} dt = \int_0^1 \frac{\partial f}{\partial z}(\gamma(t))(z - z_0) dt.$$

Consequently,

$$\begin{aligned}
\frac{f(z) - f(z_0)}{z - z_0} &= \int_0^1 \frac{\partial f}{\partial z}(\gamma(t)) dt \\
&= \int_0^1 \left(\frac{\partial f}{\partial z}(z_0) + \frac{\partial f}{\partial z}(\gamma(t)) - \frac{\partial f}{\partial z}(z_0) \right) dt \\
&= \frac{\partial f}{\partial z}(z_0) + \int_0^1 \left(\frac{\partial f}{\partial z}(\gamma(t)) - \frac{\partial f}{\partial z}(z_0) \right) dt.
\end{aligned}$$

We know $|\gamma(t) - z_0| = t|z - z_0| \leq |z - z_0|$, since $0 \leq t \leq 1$. Also, $f \in C^1$, so $\frac{\partial f}{\partial z}$ is continuous. Therefore, if $\varepsilon > 0$, there exists $\delta > 0$ such that if $|w - z| < \delta$, then $\left| \frac{\partial f}{\partial z}(w) - \frac{\partial f}{\partial z}(z_0) \right| < \varepsilon$.

Therefore, if $|z - z_0| < \delta$, then $|\gamma(t) - z_0| < \delta$, and by **Lemma 1.6.13**,

$$\left| \int_0^1 \frac{\partial f}{\partial z}(\gamma(t)) - \frac{\partial f}{\partial z}(z_0) dt \right| \leq \int_0^1 \left| \frac{\partial f}{\partial z}(\gamma(t)) - \frac{\partial f}{\partial z}(z_0) \right| dt < \varepsilon.$$

Thus, we have shown that if $|z - z_0| < \delta$,

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - \frac{\partial f}{\partial z}(z_0 \right| = \left| \int_0^1 \frac{\partial f}{\partial z}(\gamma(t)) - \frac{\partial f}{\partial z}(z_0) dt \right| < \varepsilon;$$

therefore, $f'(z_0)$ exists for all $z_0 \in U$. □

This theorem has a converse:

Theorem 1.7.4. *If $f \in C^1(U)$ and f has a complex derivative at every point of U, then f is holomorphic on U, written $f \in H(U)$.*

Proof. It suffices to show that f satisfies the **Cauchy-Riemann 1.5.4** equations. This is done by picking special paths, namely horizontal and vertical ones. See that for $z_0 \in U$,

$$\begin{aligned}
f'(z_0) &= \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} \\
&= \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} \\
&= \lim_{\substack{h\to 0\\ h\in\mathbf{R}}} \frac{f(z_0+h)-f(z_0)}{h} \\
&= \lim_{\substack{h\to 0\\ h\in\mathbf{R}}} \frac{u(x_0+h,y_0)+iv(x_0+h,y_0)-u(x_0,y_0)-iv(x_0,y_0)}{h} \\
&= \lim_{\substack{h\to 0\\ h\in\mathbf{R}}} \frac{u(x_0+h,y_0)-u(x_0,y_0)}{h} + i\lim_{\substack{h\to 0\\ h\in\mathbf{R}}} \frac{v(x_0+h,y_0)-v(x_0,y_0)}{h} \\
&= \frac{\partial u}{\partial x}(z_0) + i\frac{\partial v}{\partial x}(z_0).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
f'(z_0) &= \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} \\
&= \lim_{\substack{h\to 0\\ h\in\mathbf{R}}} \frac{f(z_0+ih)-f(z_0)}{ih} \\
&= \lim_{\substack{h\to 0\\ h\in\mathbf{R}}} \frac{u(x_0,y_0+h)+iv(x_0,y_0+h)-u(x_0,y_0)-iv(x_0,y_0)}{ih} \\
&= \frac{1}{i}\left(\frac{\partial u}{\partial y}(z_0) + i\frac{\partial v}{\partial y}(z_0)\right),
\end{aligned}$$

as before. Matching up real and imaginary terms, we have that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Finally, since $f \in C^1(U)$, so are u and v. Thus, $f \in H(U)$. □

Loosely speaking, a mapping is conformal at a point $z \in \mathbf{C}$ if the map preserves angles and stretches equally in all directions. We will talk more about conformal maps in the coming pages (see **Definition 2.1.1**), but this is a good intuition to have about the local behavior of a holomorphic function- geometrically, they are angle and scale preserving.

Theorem 1.7.5. *Let f be holomorphic in a neighborhood of $z \in \mathbf{C}$. Let $w_1, w_2 \in \mathbf{C}$ be of unit modulus; i.e., $|w_1| = |w_2| = 1$. Denote the directional derivative by*

$$D_{w_j} f(z) = \lim_{t\to 0} \frac{f(z+tw_j)-f(z)}{t},$$

for $j = 1, 2$. Then, $|D_{w_1} f(z)| = |D_{w_2} f(z)|$.

Further, the angle between w_1 and w_2 is the same as between $f'(z)w_1$ and $f'(z)w_2$, if $f'(z) \neq 0$. We could write this as $\angle w_1 w_2 = \angle Dw_1 f(z) Dw_2 f(z)$.

Proof. Observe that

$$D_{w_j} = \lim_{t\to 0} \frac{f(z+tw_j)-f(z)}{tw_j}\frac{tw_j}{t} = f'(z)w_j$$

for $j = 1, 2$. Thus, $|D_{w_j} f(z)| = |f'(z)|$, since $|w_j| = 1$.

For the angle preserving, write $w_j = e^{i\theta_j}$ and $f'(z) = re^{i\psi}$. Then the angle $\angle w_1 w_2 = \theta_2 - \theta_1$, and $\angle D_{w_1} f(z) D_{w_2} f(z) = \angle f'(z)w_1 f'(z)w_2 = \theta_2 + \psi - (\theta_1 + \psi) = \theta_2 - \theta_1$. □

1.8 Antiderivatives

Definitions:
Main Idea: Holomorphic functions on open rectangles and disks have holomorphic antiderivatives. If we're holomorphic everywhere except finitely many points (probably countably many with no accumulation point), then we have a holomorphic antiderivative.

Earlier, we saw that holomorphic functions have holomorphic antiderivatives, at least locally (**Theorem 1.5.15**). This was a consequence of constructing C^1 functions f and g such that $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$. We need a technical extension of this. In particular, we need to understand the complex line integral of $\frac{F(z)-F(z_0)}{z-z_0}$, when $F \in H(U)$ and $z_0 \in U$ are fixed.

Lemma 1.8.1. *Let $(\alpha, \beta) \subseteq \mathbf{R}$ be an open interval and $F, H : (\alpha, \beta) \to \mathbf{R}$ be continuous. Let $p \in (\alpha, \beta)$ and suppose that $\frac{dH}{dx}$ exists and equals $F(x)$ for all $x \in (\alpha, \beta) \setminus \{p\}$. Then $\frac{dH}{dx}(p)$ exists and $\frac{dH}{dx}(x) = F(x)$ for all $x \in (\alpha, \beta)$.*

Proof. It suffices to prove the result on a compact subinterval $[a, b] \subseteq (\alpha, \beta)$ where $p \in (a, b)$, since $\frac{dH}{dx} = F$ off p. This allows us to use the Fundamental Theorem of Calculus.

Set

$$K(x) = H(a) + \int_a^x F(t)dt.$$

By the Fundamental Theorem of Calculus, $K'(x)$ exists, and $K'(x) = F(x)$ on $[a, p)$ and $(p, b]$. This means there exist constants c_1 and c_2 such that

$$K(x) - H(x) = \begin{cases} c_1 & \text{for } x \in [a, p); \\ c_2 & \text{for } x \in (p, b]. \end{cases}$$

Both K and H are continuous, so

$$K(p) - H(p) = \lim_{x \to p} K(x) - H(x).$$

Thus $c_1 = c_2$. Also, $K(a) = H(a)$, so $c_1 = c_2 = 0$. Thus $K(x) = H(x)$. $\square$

Theorem 1.8.2. *Let $U \subseteq \mathbf{C}$ be either an open rectangle or an open disk, and let $p \in U$. Let $f, g \in C(U, \mathbf{R})$ and $f, g \in C^1(U \setminus \{p\})$. Suppose further that*

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$$

on $U \setminus \{p\}$. Then there exists $h \in C^1(U, \mathbf{R})$ such that

$$\frac{\partial h}{\partial x} = f \qquad \text{and} \qquad \frac{\partial h}{\partial y} = g$$

on all of U.

Proof. As before, fix $(a_0, b_0) = a_0 + ib_0 \in U$, and set

$$h(x) = \int_{a_0}^x f(t, b_0)dt + \int_{b_0}^y g(x, s)ds.$$

By the Fundamental Theorem of Calculus,

$$\frac{\partial h}{\partial y}(x, y) = g(x, y).$$

The trouble, as before, is to compute $\frac{\partial h}{\partial x}$.

First, assume that p is not on the line segment from (x, b_0) to (x, y). For such points,

$$\begin{aligned}\frac{\partial}{\partial x}\left[\int_{a_0}^{x} f(t, b_0)dt + \int_{b_0}^{y} g(x, s)ds\right] &= f(x, b_0) + \int_{b_0}^{y} \frac{\partial g}{\partial x}(x, s)ds \\ &= f(x, b_0) + \int_{b_0}^{y} \frac{\partial f}{\partial s}(x, s)ds \\ &= f(x, b_0) + f(x, y) - f(x, b_0) \\ &= f(x, y).\end{aligned}$$

For the general case, fix y and set $H(x) = h(x, y)$ and $F(x) = f(x, y)$. Then H and F are continuous, real valued, and $\frac{dH}{dx}(x) = F(x)$, except possibly at p_1, where $p = (p_1, p_2)$. Thus by **Lemma 1.8.1**, $\frac{dH}{dx}(x) = F(x)$ for all x. Hence, $\frac{\partial h}{\partial x} = f$ everywhere on U.

A similar argument shows that $\frac{\partial h}{\partial y} = g$ everywhere on U. □

The previous theorem allows us to construct holomorphic antiderivatives:

Theorem 1.8.3. *Let $U \subseteq \mathbf{C}$ be an open rectangle or disk. Fix $p \in U$ and suppose that $F \in C(U) \cap H(U \setminus \{p\})$. Then there exists $H \in H(U)$ such that $\frac{\partial H}{\partial z} = F$.*

Proof. Repeat the proof of **Theorem 1.5.15** with **Theorem 1.8.2** replacing **Theorem 1.5.13**. □

Note that **Theorem 1.8.3** will be applied extensively in the following situation (see the **Cauchy Integral Formula 1.9.3**):

> Let $f \in H(U)$ and let $z \in U$ be fixed. Set
>
> $$F(\zeta) = \begin{cases} \frac{f(\zeta)-f(z)}{\zeta - z} & \text{if } \zeta \in U \setminus \{z\}; \\ f'(z) & \text{if } \zeta = z. \end{cases}$$
>
> Since $f \in H(U)$, $F \in H(U \setminus \{z\}) \cap C(U)$.

We also remark that these results can be extended with no new ideas to handle the case of finitely many singular points, $\{p_1, ..., p_k\}$. Simply choose (a_0, b_0) so that the line segment from $(a_0, b_0$ to p_j does not contain p_ℓ for $\ell \neq j$. This is easy to do; just choose (a_0, b_0) so that it avoids the $\binom{k}{2}$ line segments formed by $\{p_1, ..., p_k\}$.

1.9 The Cauchy Integral Formula and the Cauchy Integral Theorem

Definitions: disk, piecewise C^1 curve, integral around a piecewise C^1 contour, homotopic with fixed endpoints, homotopic as closed curves, simply connected, convex

Main Idea: The Cauchy Integral Formula tells us that a holomorphic function in a disk is defined by its value on its boundary circle. The Cauchy Integral Theorem tells us that the contour integral of holomorphic functions on loops is 0. We also briefly touch on homotopies and deformation theorems. We can extend the Cauchy Integral Formula to n derivatives, which shows that derivatives of holomorphic functions are holomorphic. Finally, Morera's Theorem is a converse of sorts to Cauchy's Integral Theorem; if U is connected and open and every contour integral of f on closed curves is 0, then f is holomorphic on U.

Definition 1.9.1. The notation for **disks** will be used extensively throughout these notes. If $p \in \mathbf{C}$ and $r \in \mathbf{R}$ is positive, then

- the open disk centered at p of radius r is $D(p, r) = \{z \in \mathbf{C} \mid |z - p| < r\}$,
- the closed disk centered at p of radius r is $\overline{D(p, r)} = \{z \in \mathbf{C} \mid |z - p| \leq r\}$, and
- the circle centered at p of radius r, i.e., the boundary of the disk, is $\partial D(p, r) = \{z \in \mathbf{C} \mid |z - p| = r\}$.

We can parameterize the boundary of a disk as follows:

Let $\gamma : [0,1] \to \mathbf{C}$ be the function $\gamma(t) = p + re^{it2\pi}$, with counterclockwise, or positive, orientation. Similarly, $\sigma : [0,1] \to \mathbf{C}$, $\sigma(t) = p + re^{-it2\pi}$ has clockwise, or negative, orientation.

Lemma 1.9.2. *Let γ be the boundary of $D(z_0, r) \subseteq \mathbf{C}$ equipped with a counterclockwise orientation. Let $z \in D(z_0, r)$. Then*

$$\frac{1}{2\pi i}\oint_\gamma \frac{1}{\zeta - z}d\zeta = 1.$$

Proof. Set

$$I(z) = \oint_\gamma \frac{1}{\zeta - z}d\zeta.$$

We will prove this lemma in two steps:

1. Show that $I(z)$ is independent of $z \in D(z_0, r)$, and
2. Compute $I(z_0) = 2\pi i$.

To prove 1., start by observing that $\frac{1}{\zeta - z}$ is bounded in $\zeta \in \partial D(z_0, r)$ when z is fixed and stays away from the boundary $\partial D(z_0, r)$. Thus, we can differentiate under the integral and compute:

$$\frac{\partial}{\partial \overline{z}}I(z) = \oint_\gamma \frac{\partial}{\partial \overline{z}}\left[\frac{1}{\zeta - z}\right]d\zeta = \oint_\gamma 0 d\zeta = 0,$$

and

$$\frac{\partial}{\partial z}I(z) = \oint_\gamma \frac{\partial}{\partial z}\left[\frac{1}{\zeta - z}\right]d\zeta = \oint_\gamma \frac{1}{(\zeta - z)^2}d\zeta.$$

Since $\frac{1}{(\zeta - z)^2} = \frac{\partial}{\partial \zeta}\left[\frac{-1}{\zeta - z}\right]$ is holomorphic on $\mathbf{C} \setminus \{z\}$, by **Lemma 1.6.10**,

$$\frac{\partial}{\partial z}I(z) = \oint_\gamma -\frac{\partial}{\partial \zeta}\left[\frac{1}{\zeta - z}\right]d\zeta = -\frac{1}{\gamma(1) - z} + \frac{1}{\gamma(0) - z} = 0,$$

since $\gamma(0) = \gamma(1)$. Thus I is constant in $z \in D(z_0, r)$.

For part 2., this is a straightforward calculation. Since we are allowed to pick a curve γ, we choose $\gamma : [0, 2\pi] \to \mathbf{C}$ defined by $\gamma(t) = z_0 + re^{it}$. Then $\gamma'(t) = ire^{it}$, so

$$\oint_\gamma \frac{1}{\zeta - z_0}d\zeta = \int_0^{2\pi} \frac{1}{z_0 + re^{it} - z_0}ire^{it}dt = \int_0^{2\pi} idt = 2\pi i,$$

as desired. □

Theorem 1.9.3 (**The Cauchy Integral Formula**)**.** *Suppose $U \subseteq \mathbf{C}$ is an open set and $f \in H(U)$. Let $z_0 \in U$ and $r > 0$ such that $\overline{D(z_0, r)} \subseteq U$. Let γ be a C^1 curve that parameterizes $\partial D(z_0, r)$ counterclockwise. Then for each point $z \in D(z_0, r)$,*

$$f(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta.$$

We preceed the proof with some preemptive comments:

1. There is nothing like this formula in real analysis. For example, fix $k \in \mathbf{N}$ and let $f(x, y) = \left(1 - \left(x^2 + y^2\right)\right)^k$. Then $f(x, y) = 0$ on $\partial D(0, 1)$ while $f(x, y) > 0$ in $D(0, 1)$. Of course, $f \notin H(D(0, 1))$, since $f(x, y) = (1 - z\overline{z})^k$.

2. If f is continuous and satisfies the Cauchy Integral Formula, then f is holomorphic.

3. If f is holomorphic, then f is analytic, i.e., has a power series, and f is C^∞.

Proof. Choose $\varepsilon > 0$ so that $D(z_0, r+\varepsilon) \subseteq U$. Fix $z \in D(z_0, r+\varepsilon)$.Then $\frac{f(\zeta)}{\zeta - z}$ is holomorphic in ζ on $D(z_0, r+\varepsilon) \setminus \{z\}$. By **Theorem 1.8.3**, there exists $H \in H(D(z_0, r+\varepsilon))$ such that

$$\frac{\partial H}{\partial \zeta} = \begin{cases} \frac{f(\zeta)-f(z)}{\zeta - z} & \text{if } \zeta \neq z; \\ f'(z) & \text{if } \zeta = z. \end{cases}$$

Now, γ is a simple, closed curve, so

$$0 = H(\gamma(1)) - H(\gamma(0)) = \oint_\gamma \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta.$$

Consequently, by **Lemma 1.9.2**,

$$\oint_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta = \oint_\gamma \frac{f(z)}{\zeta - z} d\zeta = f(z) \oint_\gamma \frac{1}{\zeta - z} d\zeta = 2\pi i f(z),$$

and the Cauchy Integral Formula follows. $\square$

Theorem 1.9.4 (The Cauchy Integral Theorem). *If f is holomorphic on an open disk $U \subseteq \mathbf{C}$, and if $\gamma : [a, b] \to U$ is a C^1 curve in U with $\gamma(a) = \gamma(b)$, then*

$$\oint_\gamma f(z) dz = 0.$$

Proof. First, there is a holomorphic antiderivative of f; i.e., there exists $G \in H(U)$ such that $\frac{\partial G}{\partial z} = f$ on U. Next, since γ is closed, $\gamma(a) = \gamma(b)$, and

$$0 = G(\gamma(b)) - G(\gamma(a)) = \oint_\gamma G'(z) dz = \oint_\gamma f(z) dz,$$

since $G'(z) = \frac{\partial G}{\partial z}(z)$. $\square$

Our goal now is to explore the **Cauchy Integral Formula 1.9.3** and the **Cauchy Integral Theorem 1.9.4**. We begin with a real analysis digression. Recall

Theorem 1.9.5 (Green's Theorem). *Let $U \subseteq \mathbf{C}$ be a C^1 domain whose boundary is parameterized by a C^1 curve γ, oriented positively. If $P, Q \in C^1(U)$, then the line integral*

$$\oint_\gamma (P dx + Q dy) = \iint\limits_U \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy.$$

Let $f : \overline{U} \to \mathbf{C}$ be C^1. Write $f = u + iv$. Then

$$\oint_\gamma f dz = \oint_\gamma (u + iv)(dx + idy) = \oint_\gamma (u dx + (-v) dy) + i \oint_\gamma (v dx + u dy).$$

Applying **Green's Theorem 1.9.5** with $u = P$, $-v = Q$, we have

$$\oint_\gamma f dz = \iint\limits_U \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy + i \iint\limits_U \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx dy.$$

Also, see that

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) [u + iv] = \frac{1}{2} \left[\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) + i \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right].$$

Thus, **Green's Theorem 1.9.5** says that

$$\oint_\gamma f dz = 2i \iint\limits_U \frac{\partial f}{\partial \overline{z}} dxdy.$$

If f is holomorphic on U and γ is simple, then the **Cauchy Integral Theorem 1.9.4** follows.
We can also use the fact that

$$\oint_\gamma f dz = 2i \iint_U \frac{\partial f}{\partial \overline{z}} dxdy$$

to prove the **Cauchy Integral Formula 1.9.3**, and an inhomogeneous Cauchy Integral Formula:

Theorem 1.9.6 (The Inhomogeneous Cauchy Integral Formula). *Let $\Omega \subseteq \mathbf{C}$ be a bounded domain with C^1 boundary. If $f \in C^1(\overline{\Omega})$, then*

$$f(z) = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{\pi} \iint\limits_\Omega \frac{\frac{\partial f}{\partial \overline{\zeta}}(\zeta)}{\zeta - z} d\xi d\eta,$$

where $\zeta = \xi + i\eta$.

Proof. Fix $z \in \Omega$ and let $g(\zeta) = \frac{f(\zeta)}{\zeta - z}$. We apply the result from **Green's Theorem 1.9.5** to g on $\Omega_\varepsilon = \Omega \setminus \overline{D(z,\varepsilon)}$. Then

$$\oint_{\partial\Omega_\varepsilon} g d\zeta = \oint_{\partial\Omega} g d\zeta - \oint_{\partial D(z,\varepsilon)} g d\zeta.$$

Observe that $\partial D(z,\varepsilon)$ is parameterized by $\widetilde{\gamma}(t) = z + \varepsilon e^{it}$. Also,

$$\oint_{\partial D(z,\varepsilon)} g d\zeta = \int_0^{2\pi} \frac{f\left(z + \varepsilon e^{it}\right)}{z + \varepsilon e^{it} - z} \varepsilon i e^{it} dt = i \int_0^{2\pi} f\left(z + \varepsilon e^{it}\right) dt,$$

which goes to $2\pi i f(z)$ as $\varepsilon \to 0$, because f is continuous and $[0, 2\pi]$ is compact, so $f\left(z + \varepsilon e^{it}\right)$ converges to $f(z)$ uniformly as $\varepsilon \to 0$.

Thus, since $\frac{1}{\zeta - z}$ is integrable at z, we may send $\varepsilon \to 0$ to obtain the result. □

We now explore some examples related to the **Cauchy Integral Formula 1.9.3**.

Example 1.9.7. Let $\Omega = D(0,1)$ and $\gamma : [0, 2\pi] \to \mathbf{C}$ be defined by $\gamma(t) = e^{it} = \cos t + i \sin t$. Recall De Moivre's formula:

$$(\cos t + i \sin t)^n = \cos(nt) + i \sin(nt).$$

Next, $\frac{1}{\cos t + i \sin t} = \cos t - i \sin t$.

Furthermore, $\zeta^k \in H(\mathbf{C})$, so by the **Cauchy Integral Theorem 1.9.4**,

$$\oint_\gamma \zeta^k d\zeta = 0, \text{ if } k \geq 1.$$

But we can check this directly;

$$\begin{aligned}
\oint_\gamma \zeta^k d\zeta &= \int_0^{2\pi} (\cos t + i \sin t)^k (-\sin t + i \cos t)\, dt \\
&= i \int_0^{2\pi} (\cos t + i \sin t)^{k+1} dt \\
&= i \int_0^{2\pi} (\cos((k+1)t) + i \sin((k+1)t))\, dt \\
&= 0.
\end{aligned}$$

Also see that if $k < 0$ and $k \neq -1$, then

$$0 = i \int_0^{2\pi} (\cos t + i \sin t)^{k+1} \, dt = i \int_0^{2\pi} (\cos t - i \sin t)^{-(k+1)} \, dt.$$

Therefore, if $k \in \mathbf{Z} \setminus \{-1\}$, then

$$\oint_\gamma \zeta^k d\zeta = 0.$$

(We already know that

$$\oint_\gamma \frac{1}{\zeta - z} d\zeta = 2\pi i$$

if $z \in D(0,1)$ by **Lemma 1.9.2**, so certainly if $z = 0$.)

Example 1.9.8. Now let's examine the **Cauchy Integral Formula 1.9.3** applied to the polynomial

$$p(z) = \sum_{n=0}^N a_n z^n.$$

Since p is holomorphic,

$$p(z) = \frac{1}{2\pi i} \oint_\gamma \frac{p(\zeta)}{\zeta - z} d\zeta$$

if $|z| < 1$. Let's check this.

Formally, $\zeta \in \partial D(0,1)$ and $z \in D(0,1)$, so $|z| < |\zeta|$. So

$$\frac{1}{\zeta - z} = \frac{1}{\zeta\left(1 - \frac{z}{\zeta}\right)} = \frac{1}{\zeta} \sum_{k=0}^\infty \left(\frac{z}{\zeta}\right)^k,$$

since $|\frac{z}{\zeta}| < 1$. Thus,

$$\oint_\gamma \frac{p(\zeta)}{\zeta - z} d\zeta = \oint_\gamma \frac{p(\zeta)}{\zeta} \left(\sum_{k=0}^\infty \left(\frac{z}{\zeta}\right)^k\right) d\zeta = \oint_\gamma \sum_{k=0}^\infty \frac{p(\zeta)}{\zeta^{k+1}} z^k d\zeta$$

Dangerously commuting limits, this is equal to

$$\sum_{k=0}^\infty z^k \left(\oint_\gamma \frac{p(\zeta)}{\zeta^{k+1}} d\zeta\right) = \sum_{k=0}^\infty z^k \left[\sum_{n=0}^N a_n \oint_\gamma \zeta^{n-k-1} d\zeta\right].$$

By **Example 1.9.7**, the only term that survives is when $n - k - 1 = -1$; i.e., exactly $n = k$. Then we have

$$\sum_{k=0}^\infty z^k \left[\sum_{n=0}^N a_n \oint_\gamma \zeta^{n-k-1} d\zeta\right] = \sum_{k=0}^\infty z^k a_k (2\pi i) = 2\pi i \sum_{k=0}^N a_k z^k = 2\pi i p(z).$$

So, *formally*, indeed

$$\oint_\gamma \frac{p(\zeta)}{\zeta - z} d\zeta = 2\pi i p(z).$$

With some examples behind us, we are a little more comfortable with the **Cauchy Integral Formula 1.9.3** and **Cauchy Integral Theorem 1.9.4**. Our goal now is to examine these results for rough curves, in particular, piecewise C^1 curves, like rectangles, polygons, etc.

Definition 1.9.9. A **piecewise C^1 curve** $\gamma : [a, b] \to \mathbf{C}$, $a < b$, is a continuous function such that there exist $\{a_1, ..., a_k\} \subseteq [a, b]$ finite and satisfying $a = a_1 \leq \cdots \leq a_k = b$ such that for every $j \in \{1, ..., k-1\}$, $\gamma|_{[a_j, a_{j+1}]}$ is a C^1 curve.

Definition 1.9.10. If $U \subseteq \mathbf{C}$ and $\gamma : [a, b] \to U$ is a piecewise C^1 curve, then for $f \in C(U)$, define the **contour integral around a piecewise C^1 contour** to be

$$\oint_\gamma f(z)dz = \sum_{j=1}^{k} \oint_{\gamma|_{[a_j, a_{j+1}]}} f(z)dz.$$

We would need to check that this integral is well-defined; i.e., the sum on the right hand side does not depend on the choice of a_j or of k. (Nothing in **Definition 1.9.9** requires that the piecewise decomposition be minimal, or unique.) We would need to do this, but it's boring and technical so we won't.

The next few results all pass through to piecewise C^1 curves.

Proposition 1.9.11. *Let $\gamma : [a, b] \to \boldsymbol{C}$ be a piecewise C^1 curve, and let $U \subseteq \mathbf{C}$ be open. Let $\varphi : [c, d] \to [a, b]$ be a piecewise C^1, strictly monotonically increasing function with $\varphi(c) = a$ and $\varphi(d) = b$. If $f \in C(U)$, then $\gamma \circ \varphi : [c, d] \to U$ is a piecewise C^1 curve and*

$$\oint_\gamma f(z)dz = \oint_{\gamma \circ \varphi} f(z)dz.$$

Lemma 1.9.12. *If $f \in H(U)$ and $\gamma : [a, b] \to U$ is a piecewise C^1 curve, then*

$$f(\gamma(b)) - f(\gamma(a)) = \oint_\gamma f'(z)dz.$$

Proof. This follows immediately from the same result where γ is C^1, **Lemma 1.6.10**, and the definition of a piecewise C^1 curve, **Definition 1.9.9**. ☐

Our upcoming goal is to build a deformation theorem; establishing when integrals are independent of path. We will state it, but it will not be until the second semester notes (unfortunately) that we delve deeper into homotopic paths and topology.

Definition 1.9.13. Suppose $\gamma_0 : [a, b] \to U$ and $\gamma_1 : [a, b] \to U$ are continuous curves from z_0 to z_1 in U. We say that γ_0 **is homotopic with fixed endpoints to** γ_1 in U if there exists a homotopy, a continuous function $H : [0, 1] \times [a, b] \to U$ such that

1. $H(0, t) = \gamma_0(t)$ for $t \in [a, b]$,
2. $H(1, t) = \gamma_1(t)$ for $t \in [a, b]$,
3. $H(s, 0) = z_0$ for $s \in [0, 1]$, and
4. $H(s, 1) = z_1$ for $s \in [0, 1]$.

Definition 1.9.14. Suppose $\gamma_0, \gamma_1 : [a, b] \to U$ are continuous closed curves in U. We say γ_0 and γ_1 are **homotopic as closed curves** in U if there exists a homotopy, a continuous function $H : [0, 1] \times [a, b] \to U$ such that

1. $H(0, t) = \gamma_0(t)$ for $t \in [a, b]$,
2. $H(1, t) = \gamma_1(t)$ for $t \in [a, b]$, and
3. $H(s, 0) = H(s, 1)$ for $s \in [0, 1]$.

Definition 1.9.15. A set U is called **simply connected** if every closed curve is homotopic (as closed curves) to a point in U.

Definition 1.9.16. A set A is called **convex** if it contains the straight line segments between every pair of points. This means that if $z_0, z_1 \in A$, then so is $\{tz_1 + (1-t)z_0 \mid 0 \le t \le 1\}$.

Lemma 1.9.17. *If A is a convex region, then any two closed curves in A are homotopic as closed curves, and any two curves with the same endpoints are homotopic with fixed endpoints.*

Proof. Let $\gamma_0, \gamma_1 : [a,b] \to A$ be the two curves. Set $H(s,t) = s\gamma_1(t) + (1-s)\gamma_0(t)$. Then for any t, $H(s,t)$ lies on the line segment from $\gamma_0(t)$ to $\gamma_1(t)$, and hence is in A. Since γ_0 and γ_1 are continuous, so is H. The other properties follow quickly. $\square$

Corollary 1.9.18. *A convex region is simply connected.*

Proof. Apply **Lemma 1.9.17** with $\gamma_1(t) = \{z_0\}$ where $z_0 \in A$. $\square$

Proposition 1.9.19 (Deformation Theorem). *Suppose $U \subseteq \mathbf{C}$ and $f \in H(U)$. Let γ_0, γ_1 be piecewise C^1 curves in U.*

1. *If γ_0 and γ_1 are paths from z_0 to z_1 and are homotopic in U with fixed endpoints, then*
$$\oint_{\gamma_0} f dz = \oint_{\gamma_1} f dz.$$
2. *If γ_0 and γ_1 are closed curves that are homotopic as closed curves in U, then*
$$\oint_{\gamma_0} f dz = \oint_{\gamma_1} f dz.$$

As stated, we don't delve into the proof here. However, it is intuitively straightforward; if γ_0 and γ_1 are homotopic with fixed endpoints, then their composition is a closed curve and thus by the **Cauchy Integral Theorem 1.9.4** integrates to 0. For 2., one can traverse along contours between γ_0 and γ_1 that cancel each other out. See next semester.

We now explore a few application of the **Cauchy Integral Formula 1.9.3** and **Cauchy Integral Theorem 1.9.4**.

Theorem 1.9.20. *Let $U \subseteq \mathbf{C}$ be an open set, and $f \in H(U)$. Then $f \in C^\infty(U)$. Moreover, if $\overline{D(z_0, r)} \subseteq U$ and $z \in D(z_0, r)$, then*
$$\frac{\partial^k f}{\partial z^k}(z) = \frac{k!}{2\pi i} \oint_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta$$
for $k \in \boldsymbol{W}$.

Proof. For $z \in D(z_0, r)$, the function $\zeta \mapsto \frac{f(\zeta)}{\zeta - z}$ is continuous on $\partial D(z_0, r)$. Also, for $\zeta \in \partial D(z_0, r)$,
$$|\zeta - z| = |\zeta - z_0 + z_0 - z| \ge |\zeta - z_0| - |z_0 - z| = r - |z_0 - z| > 0.$$
Hence, $\frac{f(\zeta)}{\zeta - w}$ converges to $\frac{f(\zeta)}{\zeta - z}$ as $w \to z$, uniformly in $\zeta \in \partial D(z_0, r)$, as $\partial D(z_0, r)$ is compact. This means that the difference quotient
$$\frac{1}{h}\left(\frac{f(\zeta)}{\zeta - (z+h)} - \frac{f(\zeta)}{\zeta - z}\right) = \frac{f(\zeta)}{h}\left(\frac{\zeta - z - (\zeta - (z+h))}{(\zeta - (z+h))(\zeta - z)}\right) = \frac{f(\zeta)}{(\zeta - (z+h))(\zeta - z)}$$
converges to $\frac{f(\zeta)}{(\zeta - z)^2}$ uniformly in $\zeta \in \partial D(z_0, r)$ as $h \to 0$. Therefore, by advanced calculus,
$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \frac{1}{2\pi i} \cdot \frac{1}{h} \oint_{|\zeta - z_0| = r} \left(\frac{f(\zeta)}{\zeta - (z+h)} - \frac{f(\zeta)}{\zeta - z}\right) d\zeta,$$

which, by uniform convergence, is equal to

$$\frac{1}{2\pi i}\oint_{|\zeta-z_0|=r}\lim_{h\to 0}\frac{1}{h}\left(\frac{f(\zeta)}{\zeta-(z+h)}-\frac{f(\zeta)}{\zeta-z}\right)d\zeta=\frac{1}{2\pi i}\oint_{|\zeta-z_0|=r}\frac{f(\zeta)}{(\zeta-z)^2}d\zeta.$$

As a consequence of this calculation, we have that

$$f'(z)=\frac{1}{2\pi i}\oint_{|\zeta-z_0|=r}\frac{f(\zeta)}{(\zeta-z)^2}d\zeta.$$

We can repeat the argument, with $\frac{f(\zeta)}{(\zeta-z)^2}$ replacing $\frac{f(\zeta)}{\zeta-z}$, to show that f' itself has a complex derivative at each point of $D(z_0,r)$, and

$$f''(z)=(f'(z))'=\frac{2}{2\pi i}\oint_{|\zeta-z_0|=r}\frac{f(\zeta)}{(\zeta-z)^3}d\zeta.$$

Finally, since $\frac{f(\zeta)}{(\zeta-w)^3}\to\frac{f(\zeta)}{(\zeta-z)^3}$ as $w\to z$ uniformly in $\zeta\in\partial D(z_0,r)$, it follows that $f''\in C(D(z_0,r))$ and hence f' is $C^1(D(z_0,r))$, hence holomorphic on $D(z_0,r)$.

Repeating this argument $k-1$ times shows that $f^{(k)}\in H(D(z_0,r))$ and $f\in C^{k+1}(D(z_0,r))$. Thus, $f\in C^\infty(D(z_0,r))$. □

Corollary 1.9.21. *If $f\in H(U)$, then $f'\in H(U)$.*

Observe that the argument in the proof of **Theorem 1.9.20** actually has proved the following:

Theorem 1.9.22. *If φ is a continuous function on $\{\zeta\mid|\zeta-z_0|=r\}$, then the function f defined by*

$$f(z)=\frac{1}{2\pi i}\oint_{|\zeta-z_0|=r}\frac{\varphi(\zeta)}{\zeta-z}d\zeta$$

is defined and holomorphic on $D(z_0,r)$.

Thus the **Cauchy Integral Formula 1.9.3** can take in a continuous function and spit out a holomorphic one.

Example 1.9.23. Let $z_0=0$ and $r=1$. Choose $\varphi(\zeta)=\overline{\zeta}$. Let $\gamma(t)=e^{it}$, $0\le t\le 2\pi$. Then

$$f(z)=\frac{1}{2\pi i}\int_0^{2\pi}\frac{e^{-it}}{e^{it}-z}ie^{it}dt=\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{e^{it}}\frac{1}{1-\frac{z}{e^{it}}}dt=\frac{1}{2\pi}\int_0^{2\pi}e^{-it}\sum_{j=0}^{\infty}\left(ze^{-it}\right)^j dt=\frac{1}{2\pi}\sum_{j=0}^{\infty}z^j\int_0^{2\pi}e^{-i(j+1)t}dt=0.$$

We commute the limits by uniform convergence of the geometric series in j and in t.

It turns out that the **Cauchy Integral Theorem 1.9.4** also gives a characterization of holomorphicity.

Theorem 1.9.24 (Morera's Theorem). *Suppose that $f:U\to\mathbf{C}$ is a continuous function on a connected, open set $U\subseteq\mathbf{C}$. Assume that for every closed, piecewise C^1 curve $\gamma:[0,1]\to U$, it holds that*

$$\oint_\gamma f(\zeta)d\zeta=0.$$

Then $f\in H(U)$.

Proof. Fix $z_0\in U$. Define a function $F:U\to\mathbf{C}$ as follows: given $w\in\mathbf{C}$, choose a piecewise C^1 path $\psi:[0,1]\to U$ so that $\psi(0)=z_0$ and $\psi(1)=w$. This is possible, as U connected and open implies that U is path connected. Then set

$$F(w)=\oint_\psi f(\zeta)d\zeta.$$

We claim that F is well-defined. Let τ be another piecewise C^1 path from z_0 to w. Let γ be the curve traced along ψ, and in reverse along τ (call this curve $-\tau$). Then by hypothesis,

$$0 = \oint_\gamma f d\zeta = \oint_\psi f d\zeta + \oint_{-\tau} f d\zeta = \oint_\psi f d\zeta - \oint_\tau f d\zeta,$$

and indeed F is well-defined.

Next, we show that $F \in C^1(U)$ and satsifies the **Cauchy-Riemann 1.5.4** equations. Fix $z = x + iy \in U$ and let ψ be a piecewise C^1 curve from z_0 to z. Choose $h \in \mathbf{R}$ small enough so that $z + h \in U$. Then let the line segment connected z to $z + h$ be parameterized by $\ell_h(t) = (x + t, y)$, $t \in [0, h]$. Denote by ψ_h the piecewise C^1 curve connected z_0 to $z + h$ be ψ, followed by ℓ_h. Then

$$F(x+h, y) - F(x, y) = \oint_{\psi_h} f(\zeta) d\zeta - \oint_\psi f(\zeta) d\zeta = \oint_{\ell_h} f(\zeta) d\zeta = \int_0^h f(z+s) ds.$$

Decompose $F = U + iV$. Then

$$\frac{U(x+h, y) - U(x, y)}{h} = \frac{1}{h} \operatorname{Re} \int_0^h f(z+s) ds = \frac{1}{h} \int_0^h \operatorname{Re} f(z+s) ds,$$

which is an average value integral. So by the continuity of f at z, we may conclude that $\frac{\partial U}{\partial x}(z)$ exists, and $\frac{\partial U}{\partial x}(z) = \operatorname{Re} f(z)$.

Similarly, we can repeat the argument and conclude that $\frac{\partial U}{\partial y}(z) = -\operatorname{Im} f(z)$, $\frac{\partial V}{\partial x}(z) = \operatorname{Im} f(z)$, and $\frac{\partial V}{\partial y}(z) = \operatorname{Re} f(z)$. So the **Cauchy-Riemann 1.5.4** equations are satisfied.

Since f is continuous, so are the partials of U and V. It then follows that $F \in H(U)$. Thus, by **Theorem 1.9.20** and **Corollary 1.9.21**, $F \in C^\infty(U)$, and $f = F'$ is therefore holomorphic on U as well. $\square$

1.10 Complex Power Series

Definitions: sequence, sequence convergence, complex power series, partial sum, radius of convergence, uniform convergence

Main Idea: Holomorphic functions are analytic. An analytic function is holomorphic in its disk of convergence. Taylor series expansions are unique. The coefficients of a Taylor series expansion are $\frac{1}{k!}\frac{\partial^k f}{\partial z^k}(z_0)$.

Our goal is to show that holomorphic functions have convergent Taylor series; i.e., are analytic. In one variable, C^∞ alone does *not* imply a convergent Taylor series; a common example is

$$f(x) = \begin{cases} e^{-x^{-2}} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

A natural question to ask would be: why do we even expect this? Why might we guess that holomorphic functions have Taylor series?

Suppose $\gamma(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$, and suppose f is holomorphic on a neighborhood of $\overline{D(z_0, r)}$. Then, *formally*,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f\left(z_0 + re^{it}\right)}{re^{it} - (z - z_0)} ire^{it} dt \\ &= \frac{1}{2\pi i} \int_0^{2\pi i} \int_0^{2\pi} \frac{f\left(z_0 + re^{it}\right)}{re^{it}\left(1 - \frac{z - z_0}{re^{it}}\right)} ire^{it} dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} f\left(z_0 + re^{it}\right) \sum_{j=0}^\infty \left(\frac{z - z_0}{re^{it}}\right)^j dt \end{aligned}$$

Commuting limits leads to a highly suspect equality:

$$\frac{1}{2\pi}\int_0^{2\pi} f\left(z_0+re^{it}\right)\sum_{j=0}^{\infty}\left(\frac{z-z_0}{re^{it}}\right)^j dt=\sum_{j=0}^{\infty}\left(\frac{1}{2\pi}\int_0^{2\pi} f\left(z_0+re^{it}\right)e^{-ijt}dtr^{-j}\right)(z-z_0)^j\,.$$

Observe that there are no $\overline{z}$ terms.

Definition 1.10.1. A **sequence** of complex numbers is a function from $\{1,2,3,...\}$ to $\mathbf{C}$, or from $\{0,1,2,...\}$ to $\mathbf{C}$. We usually write $(a_1,a_2,a_3,...)$ or $(a_0,a_1,a_2,...)$.

Definition 1.10.2. We say (a_n) converges to ℓ and write $\lim\limits_{n\to\infty} a_n=\ell$ if for all $\varepsilon>0$, there exists $N=N_\varepsilon>0$ so that if $n\geq N$, then $|a_n-\ell|<\varepsilon$.

Lemma 1.10.3 (Cauchy Criterion). *Let (a_k) be a sequence of complex numbers. Then (a_k) converges if and only if for each $\varepsilon>0$, there exists $N>0$ so that if $j,k\geq N$, then $|a_j-a_k|<\varepsilon$.*

Proof. The proof is a consequence of the Cauchy criterion for real sequences. If

$$\lim_{k\to\infty} a_k=\ell,$$

then given $\varepsilon>0$, there exists $N=N(\varepsilon)$ such that if $n\geq N$, $|a_n-\ell|<\frac{\varepsilon}{2}$. Therefore, if $m,n\geq N$,

$$|a_m-a_n|\leq|a_m-\ell|+|\ell-a_n|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Now, conversely, if (a_k) satisifies the Cauchy criterion, then decompose a_k into $\alpha_k+i\beta_k$, where we have $(\alpha_k),(\beta_k)\subseteq\mathbf{R}$. Then (α_k) and (β_k) are Cauchy sequences in $\mathbf{R}$, since $|\alpha_j-\alpha_k|\leq|a_j-a_k|$ and $|\beta_j-\beta_k|\leq a_j-a_k|$. Thus, there exist $\alpha,\beta\in\mathbf{R}$ such that

$$\lim_{n\to\infty}\alpha_n=\alpha \qquad\text{and}\qquad \lim_{n\to\infty}\beta_n=\beta.$$

Consequently, $a_k=\alpha_k+i\beta_k\to\alpha+i\beta$ as $k\to\infty$. ☐

Definition 1.10.4. Let $z_0\in\mathbf{C}$ be fixed. A **complex power series** (centered at z_0) is an expression of the form

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k,$$

where (a_k) are complex constants.

Definition 1.10.5. The Nth **partial sum** of the power series, S_N, is defined as

$$S_N(z)=\sum_{j=0}^{N} a_j(z-z_0)^j.$$

We say the series converges to a limit $S(z)$ at z if $S_N(z)\to S(z)$ as $N\to\infty$.
From the Cauchy criterion,

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k$$

converges at z if and only if for each $\varepsilon>0$, there exists $N>0$ so that if $m\geq j\geq N$, then

$$\left|\sum_{k=j}^{m} a_k(z-z_0)^k\right|<\varepsilon.$$

Lemma 1.10.6 (**Abel**). *If*

$$\sum_{k=0}^{\infty}(z-z_0)^k$$

converges at some z, then the series converges at each $w \in D(z_0, r)$, where $r = |z - z_0|$.

Proof. Since

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k$$

converges, it follows that

$$\lim_{k\to\infty} a_k(z-z_0)^k = 0.$$

This forces $\left(a_k(z-z_0)^k\right)$ to be a bounded sequence; i.e., there exists $M > 0$ so that $\left|a_k(z-z_0)^k\right| \leq M$ for all $k \in \mathbf{W}$.

Since $|z - z_0| = r$, we have $\left|a_k r^k\right| \leq M$. Choose $w \in D(z_0, r)$. Then $|w - z_0| < r$, and

$$\left|a_k(w-z_0)^k\right| = |a_k| r^k \left|\frac{w-z_0}{r}\right|^k \leq M \left|\frac{w-z_0}{r}\right|^k.$$

This means that

$$\sum_{k=0}^{\infty}\left|a_k(w-z_0)^k\right| \leq M \sum_{k=0}^{\infty}\left|\frac{w-z_0}{r}\right|^k = \frac{M}{1-\frac{|w-z_0|}{r}},$$

so

$$\sum_{k=0}^{\infty} a_k(w-z_0)^k$$

converges absolutely, and hence converges. $\square$

Abel's Lemma 1.10.6 shows that the interior of the domain of convergence of a power series is a disk. This motivates the following:

Definition 1.10.7. Let

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k$$

be a power series. Set

$$r = \sup\left\{|w-z_0| \mid \sum_{k=0}^{\infty} a_k(w-z_0)^k \text{ converges.}\right\}$$

Then r is called the **radius of convergence**, and $D(z_0, r)$ is the disk of convergence.

We can thus reformulate **Abel's Lemma 1.10.6**:

Lemma 1.10.8 (**Abel**). *If*

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k$$

is a power series with radius of convergence r, then the series converges for each $w \in D(z_0, r)$, and diverges at every w such that $|w - z_0| > r$.

As a matter of notation, if $r = \infty$, then $D(z_0, r) = \mathbf{C}$, and if $r = 0$, $D(z_0, r) = \emptyset$.

Lemma 1.10.9 (The Root Test). *The radius of convergence of the power series*

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k \text{ is } r = \frac{1}{\limsup_{k\to\infty} |a_k|^{\frac{1}{k}}} \text{ if } \limsup_{k\to\infty} |a_k|^{\frac{1}{k}} > 0, \text{ and infinite if } \limsup_{k\to\infty} |a_k|^{\frac{1}{k}} = 0.$$

Proof. We first assume that

$$\alpha = \limsup_{k\to\infty} |a_k|^{\frac{1}{k}} > 0.$$

If $|z - z_0| > \frac{1}{\alpha}$, then for some $c > 1$, $|z - z_0| = \frac{c}{\alpha}$. It then must be the case that for infinitely many k, $|a_k|^{\frac{1}{k}} > \frac{\alpha}{c}$. For such k,

$$\left|a_k (z - z_0)^k\right| = \left(|a_k|^{\frac{1}{k}}\right)^k |z - z_0|^k > \left(\frac{\alpha}{c} \cdot \frac{c}{\alpha}\right)^k = 1.$$

Thus,

$$\lim_{k\to\infty} \left|a_k (z - z_0)^k\right| \neq 0,$$

so the series

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k$$

diverges.

Next, assume $|z - z_0| < \frac{1}{\alpha}$. Then there exists $d < 1$ so that $|z - z_0| = \frac{d}{\alpha}$. Choose c so that $d < c < 1$. It then follows that there exists K so that if $k \geq K$, then $|a_k|^{\frac{1}{k}} < \frac{\alpha}{c}$. For such k,

$$\left|a_k (z - z_0)^k\right| = \left(|a_k|^{\frac{1}{k}} |z - z_0|\right)^k < \left(\frac{\alpha}{c} \cdot \frac{d}{\alpha}\right)^k = \left(\frac{d}{c}\right)^k.$$

Since $\frac{d}{c} < 1$,

$$\sum_{k=0}^{\infty} \left(\frac{d}{c}\right)^k$$

converges, hence

$$\sum_{k=0}^{\infty} a_k |z - z_0|^k$$

converges absolutely, and hence converges. ∎

Convergence of power series inside the disk of convergence is much better than absolute convergence.

Definition 1.10.10. A series

$$\sum_{k=0}^{\infty} f_k(z)$$

of functions $f_k(z)$ **converges uniformly** on a set E to the function $g(z)$ if for each $\varepsilon > 0$, there exists $N_0 = N_0(\varepsilon)$ such that if $N \geq N_0$, then

$$\left|\sum_{k=0}^{N} f_k(z) - g(z)\right| < \varepsilon$$

for all $z \in E$.

We record the following for posterity; this was proved in the proof of **Abel's Lemma 1.10.6**:

Proposition 1.10.11. *Let*

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k$$

be a power series with radius of convergence r. Then for any number R with $0 \leq R < r$, the series

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k$$

converges absolutely and uniformly on $\overline{D(z_0,R)}$. The series converges absolutely and uniformly on compact subsets of $D(z_0,r)$.

Lemma 1.10.12. *If a power series*

$$\sum_{j=0}^{\infty} a_j(z-z_0)^j$$

has radius of convergence r, then the series defines a C^∞ function $f(z)$ on $D(z_0,r)$. The function f is holomorphic on $D(z_0,r)$, and the series obtained by term-by-term differenentiation

$$\sum_{j=k}^{\infty} j(j-1)\cdots(j-k+1)a_j(z-z_0)^{j-k}$$

converges on $D(z_0,r)$ to $\frac{\partial^k f}{\partial z^k}(z)$ for each $z \in D(z_0,r)$.

Proof. We first show that f has a complex derivative. With z fixed, $(f(z+h)-f(z))\frac{1}{h}$ is a difference quotient. Then

$$\frac{f(z+h)-f(z)}{h} = \sum_{j=0}^{\infty} \frac{1}{h}\left(a_j(z-z_0+h)^j - a_j(z-z_0)^j\right).$$

Next, since $z \mapsto (z-z_0)^j$ is holomorphic, and $\frac{\partial}{\partial z}\left[z^j\right] = jz^{j-1}$, if $\gamma(t) = z+th-z_0$, $t \in [0,1]$, then $\gamma'(t) = h$ and

$$a_j(z-z_0+h)^j - a_j(z-z_0)^j = a_j \oint_\gamma j(z-z_0)^{j-1}dz = a_j \int_0^1 hj(z-z_0+th)^{j-1}dt.$$

This means that

$$\left|\frac{1}{h}\left(a_j(z-z_0+h)^j - a_j(z-z_0^j)\right)\right| \leq |a_j|j\int_0^1 |z-z_0+th|^{j-1}dt \leq |a_j|j\left(|z-z_0|+|h|\right)^{j-1}.$$

In particular, if $|h| \leq \frac{1}{2}(r-|z-z_0|)$, then

$$\left|\frac{1}{h}\left(a_j(z-z_0+h)^j - a_j(z-z_0)^j\right)\right| \leq j|a_j|\left(\frac{1}{2}(r+|z-z_0|)\right)^{j-1}.$$

By the **Root Test 1.10.9**,

$$\sum_{j=0}^{\infty} j|a_j|\left(\frac{1}{2}(r+|z-z_0|)\right)^{j-1}$$

converges. By the Weierstrass M test, the series

$$\sum_{j=0}^{\infty} \frac{1}{h} \left(a_j(z - z_0 + h)^j - a_j(z - z_0)^j\right)$$

converges uniformly in h, for h small. Hence, the sum and the limit of the difference quotients commute, and $f'(z)$ exists and has the power series expansion

$$f'(z) = \sum_{j=1}^{\infty} j a_j (z - z_0)^{j-1}.$$

By the **Root Test 1.10.9**, the power series for f' has the same radius of convergence. The higher complex derivatives are obtained by induction.

Repeating the differentiation argument except with no $\frac{1}{h}$ factor will prove continuity of f.

Repeating the differentiation argument with h real shows that the $\frac{\partial}{\partial x}$ derivative exists, and with ih, $h \in \mathbf{R}$, shows that the $\frac{\partial}{\partial y}$ derivative exists. And f' is continuous by the same argument of the continuity of f applied to $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. Thus $f \in H(D(z_0, r))$. □

Lemma 1.10.13. *If both series*

$$\sum_{j=0}^{\infty} a_j (z - z_0)^j \qquad \textit{and} \qquad \sum_{j=0}^{\infty} b_j (z - z_0)^j$$

converge on $D(z_0, r)$, $r > 0$, and if

$$\sum_{j=0}^{\infty} a_j (z - z_0)^j = \sum_{j=0}^{\infty} b_j (z - z_0)^j$$

for all $z \in D(z_0, r)$, then $a_j = b_j$ for all $j \geq 0$.

Proof. Let

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j.$$

From **Lemma 1.10.12**, $f \in C^\infty(D(z_0, r))$ and

$$\frac{\partial^k f}{\partial z^k}(z) = \sum_{j=k}^{\infty} j(j-1) \cdots (j-k+1) a_j (z - z_0)^{j-k}.$$

Plugging in $z = z_0$ shows that $\frac{\partial^k f}{\partial z^k}(z_0) = k! a_k$. Since f also has the power series expansion

$$f(z) = \sum_{j=0}^{\infty} b_j (z - z_0)^j,$$

the same argument shows that $b_k = \frac{1}{k!} \frac{\partial^k f}{\partial z^k}(z_0)$ for all $k \in \mathbf{W}$. Thus, $a_k = b_k$ for all $k \in \mathbf{W}$. □

From the proof of **Lemma 1.10.13**, we observe that if f has a power series expansion on $D(z_0, r)$, then

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\partial^k f}{\partial z^k}(z_0)(z - z_0)^k.$$

Theorem 1.10.14. *Let $U \subseteq \mathbf{C}$ be an open set and $f \in H(U)$. Let $z_0 \in U$ and suppose $D(z_0, r) \subseteq U$. Then the complex power series*

$$\sum_{k=0}^{\infty} \frac{1}{k!} \frac{\partial^k f}{\partial z^k}(z_0)(z - z_0)^k$$

has radius of convergence at least r. Moreover, the series converges to $f(z)$ on $D(z_0, r)$.

Proof. We know from **Theorem 1.9.20** that $f \in C^\infty(U)$, so there is no problem in discussing $\frac{\partial^k f}{\partial z^k}(z_0)$.

Let $z \in D(z_0, r)$ and choose r' so that $|z - z_0| < r' < r$. Then $z \in D(z_0, r') \subseteq \overline{D(z_0, r')} \subseteq D(z_0, r)$. For simplicity, but no loss of generality, we may take $z_0 = 0$.

Applying the **Cauchy Integral Formula 1.9.3** to f on $D(0, r')$, we know that

$$f(z) = \frac{1}{2\pi i} \oint_{|\zeta| = r'} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \oint_{|\zeta| = r'} \frac{f(\zeta)}{\zeta} \frac{1}{1 - \frac{z}{\zeta}} d\zeta.$$

Since $|\zeta| = r'$ and $|z| < r'$, we have

$$f(z) = \frac{1}{2\pi i} \oint_{|\zeta| = r'} \frac{f(\zeta)}{\zeta} \sum_{k=0}^{\infty} \left(\frac{z}{\zeta}\right)^k d\zeta.$$

The series

$$\sum_{k=0}^{\infty} \left(\frac{z}{\zeta}\right)^k$$

converges uniformly on $\{\zeta \mid |\zeta| = r'\}$, so we may interchange the sum and the integral. Thus, by uniform convergence,

$$f(z) = \frac{1}{2\pi i} \sum_{k=0}^{\infty} z^k \oint_{|\zeta| = r'} \frac{f(\zeta)}{\zeta^{k+1}} d\zeta = \sum_{k=0}^{\infty} z^k \frac{1}{k!} \frac{\partial^k f}{\partial z^k}(0),$$

by **Theorem 1.9.20** $\square$

The following example shows that sometimes, **Theorem 1.10.14** is the best behavior you can hope for.

Example 1.10.15. Let $f(z) = \frac{1}{z - 4i}$. Let $U = \mathbf{C} \setminus \{4i\}$. At $z_0 = 0$, $f(z_0) = \frac{1}{z_0 - 4i} = \frac{-1}{4i} \cdot \frac{1}{1 - \frac{z}{4i}}$. If $|z| < 4$, we know that

$$\frac{1}{z_0 - 4i} = \frac{-1}{4i} \cdot \sum_{k=0}^{\infty} \left(\frac{z}{4i}\right)^k.$$

If the radius of convergence were bigger than 4, then f would be bounded and well-defined at $z = 4i$.

If $z_0 = 3$, then

$$f(z) = \frac{1}{z - 4i} = \frac{1}{z - 3 + (3 - 4i)} = \frac{1}{3 - 4i} \cdot \frac{1}{1 + \frac{z - 3}{3 - 4i}} = \frac{1}{3 - 4i} \cdot \sum_{k=0}^{\infty} (-1)^k \left(\frac{z - 3}{3 - 4i}\right)^k,$$

which converges exactly when $|z - 3| < |3 - 4i| = 5$.

1.11 Cauchy Estimates and Results

Definitions: entire

Main Idea: The Cauchy Estimates put bounds on the kth derivative of a holomorphic function. They're used to prove Liouville and its generalizations and the Fundamental Theorem of Algebra and its corollaries (among other things).

Theorem 1.11.1 (The Cauchy Estimates). *Let $U \subseteq \mathbf{C}$ be open and $f \in H(U)$. Let $z_0 \in U$ and assume $\overline{D(z_0, r)} \subseteq U$ for some $r > 0$. Set $M = \sup_{z \in \overline{D(z_0,r)}} |f(z)|$. Then for all $k \in \mathbf{N}$, we have that*

$$\left| \frac{\partial^k f}{\partial z^k}(z_0) \right| \leq \frac{Mk!}{r^k}.$$

Proof. By **Theorem 1.9.20**,

$$\frac{\partial^k f}{\partial z^k}(z_0) = \frac{k!}{2\pi i} \oint_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta.$$

By **Lemma 1.6.14**,

$$\left| \frac{\partial^k f}{\partial z^k}(z_0) \right| \leq \frac{k!}{2\pi} \oint_{|\zeta - z_0| = r} \left| \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} \right| d\zeta \leq \frac{k!}{2\pi r^{k+1}} \int_0^{2\pi} M \left| ire^{it} \right| dt = \frac{Mk!}{r^k},$$

as desired. □

Note that we have proven the **Cauchy Estimates 1.11.1** with $M = \sup_{z \in \partial D(z_0,r)} |f(z)|$, but with the **Maximum Modulus Theorem 1.17.16**, these will be shown to be equivalent.

The **Cauchy Estimates 1.11.1** give us a way to estimate the radius of convergence. Consider $f \in H(U)$; then

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k.$$

If $\overline{D(z_0, r)} \subseteq U$, then

$$\limsup_{k \to \infty} \left| \frac{f^{(k)}(z_0)}{k!} \right|^{\frac{1}{k}} \leq \left(\left(\frac{Mk!}{r^k} \right) \frac{1}{k!} \right)^{\frac{1}{k}} = \frac{1}{r}.$$

This means that the radius of convergence is at least r.

Lemma 1.11.2. *Suppose f is holomorphic on a connected, open set $U \subseteq \mathbf{C}$. If $\frac{\partial f}{\partial z} \equiv 0$ on U, then f is constant on U.*

Proof. Since $f \in H(U)$, $\frac{\partial f}{\partial \bar{z}} = 0$. By hypothesis, $\frac{\partial f}{\partial z} = 0$. Therefore, $\nabla f \equiv 0$ on U, and f is identically constant, since U is connected. □

Definition 1.11.3. A function $f \in H(\mathbf{C})$ is called **entire**.

Theorem 1.11.4 (Liouville's Theorem). *A bounded, entire function is constant.*

Proof. Let f be entire, and suppose $|f(z)| \leq M$ for all $z \in \mathbf{C}$ and some constant $M > 0$. Let $r > 0$ and $z_0 \in \mathbf{C}$. By the **Cauchy Estimates 1.11.1**,

$$\left| \frac{\partial f}{\partial z}(z_0) \right| \leq \frac{M}{r}.$$

M does not depend on r, and $r > 0$ was arbitrary. So it follows that $\frac{\partial f}{\partial z}(z_0) = 0$. But $z_0 \in \mathbf{C}$ was arbitrary, so $\frac{\partial f}{\partial z} \equiv 0$. By **Lemma 1.11.2**, f is constant. □

Theorem 1.11.5. *Let f be entire and suppose there exists $C > 0$ and $k \in \boldsymbol{N}$ such that $|f(z)| \leq C|z|^k$ for all $|z| \geq 1$. Then f is a polynomial in z of degree at most k.*

(Essentially, if f grows like a polynomial or slower, then it is a polynomial. This is a generalization of **Liouville's Theorem 1.11.4**; Liouville is the $k = 0$ case.)

Proof. Since f is entire, the power series for f converges for all $z \in \mathbf{C}$. Thus, it suffices to show that $\frac{\partial^{k+\ell} f}{\partial z^{k+\ell}}(0) = 0$ whenever $\ell \geq 1$.

Let $\ell \geq 1$ and fix $r > 1$. Then

$$\left|\frac{\partial^{k+\ell} f}{\partial z^{k+\ell}}(0)\right| = \frac{(k+\ell)!}{2\pi}\left|\oint_{|z|=r} \frac{f(z)}{z^{k+\ell+1}} dz\right| \leq \frac{(k+\ell)!}{2\pi}\int_0^{2\pi} \frac{Cr^{k+1}}{r^{k+\ell+1}} dt = \frac{C(k+\ell)!}{r^\ell}.$$

As in the proof of **Liouville 1.11.4**, $r > 1$ was arbitrary, so $\frac{\partial^{k+\ell} f}{\partial z^{k+\ell}} = 0$ for all $\ell \geq 1$. □

Theorem 1.11.6 (**The Fundamental Theorem of Algebra**). *Let $p(z)$ be a nonconstant holomorphic polynomial. Then p has a root; i.e., there exists $\alpha \in \mathbf{C}$ such that $p(\alpha) = 0$.*

Proof. Suppose not. Then $g(z) = \frac{1}{p(z)}$ is entire. Also, when $|z| \to \infty$, $|p(z)| \to \infty$. Thus $\frac{1}{p(z)} \to 0$ as $|z| \to \infty$; by **Liouville's Theorem 1.11.4**, g is constant, hence, p is constant. A contradiction yields the desired result. □

Corollary 1.11.7. *If $p(z)$ is a holomorphic polynomial of degree k, then there are k complex numbers $\alpha_1, ..., \alpha_k$, not necessarily distinct, and a nonzero constant $C \in \mathbf{C}$ such that $p(z) = C(z - \alpha_1)\cdots(z - \alpha_k)$.*

Proof. This follows from the **Fundamental Theorem of Algebra 1.11.6**. In particular, if p is a polynomial of degree k with root α_1, we may use polynomial long division to factor $p(z) = (z - \alpha_1)p_1(z)$, where $p_1(z)$ is a holomorphic polynomial of degree $k - 1$. Repeat. □

1.12 Uniform Limits of Holomorphic Functions

Definitions:
Main Idea: The normal limit of holomorphic functions is holomorphic. Moreover, the normal limit of the derivatives of holomorphic functions is the derivatives of the normal limit.

From advanced calculus, we know that the uniform limit of continuous functions is continuous, but the uniform limit of *smooth* functions need not be better than continuous.

Example 1.12.1. Let $f_n(x) = \frac{\sin(nx)}{\sqrt{n}}$. Then $f_n \to 0$ uniformly, but ${f_n}'(x) = \sqrt{n}\cos(nx)$ diverges as $n \to \infty$ for all x.

Of course, as is to be expected any time we compare the behavior in $\mathbf{R}$ to the behavior in $\mathbf{C}$, $\mathbf{C}$ is much nicer.

Theorem 1.12.2. *Let $U \subseteq \mathbf{C}$ be an open set and $f_j \in H(U)$, $j \in \mathbf{N}$. Suppose there exists a function $f : U \to \mathbf{C}$ such that for each compact set $E \subseteq U$, the sequence $f_j|_E$ converges uniformly to $f|_E$. Then $f \in H(U)$.*

Proof. Let $z_0 \in U$ and choose $r > 0$ so that $\overline{D(z_0, r)} \subseteq U$. This closed disk is compact, so $f_j \to f$ uniformly on $\overline{D(z_0, r)}$. Therefore, f is continuous on $\overline{D(z_0, r)}$. Thus, for any $z \in D(z_0, r)$,

$$f(z) = \lim_{j\to\infty} f_j(z) = \lim_{j\to\infty} \frac{1}{2\pi i}\oint_{|\zeta - z_0|=r} \frac{f_j(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i}\oint_{|\zeta - z_0|=r} \lim_{j\to\infty} \frac{f_j(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i}\oint_{|\zeta - z_0|=r} \frac{f(\zeta)}{\zeta - z} d\zeta,$$

by uniform convergence in $\zeta \in \partial D(z_0, r)$ for each fixed $z \in D(z_0, r)$ of $\frac{f_j(\zeta)}{\zeta - z} \to \frac{f(\zeta)}{\zeta - z}$. Since f is continuous and satsifying the **Cauchy Integral Formula 1.9.3**, by **Theorem 1.9.22**, $f \in H(D(z_0, r))$. As z_0 was arbitrary, $f \in H(U)$. □

Corollary 1.12.3. *If f_j, f, and U are as in the hypotheses of **Theorem 1.12.2**, then for any $k \in \mathbf{W}$, we have $\frac{\partial^k f_j}{\partial z^k} \to \frac{\partial^k f}{\partial z^k}$ uniformly on compact subsets of U.*

Proof. We could simply use the argument in the proof of **Theorem 1.12.2**. But notice that we can also use **Cauchy Estimates 1.11.1**:

Let $E \subseteq U$ be compact. Then there exists $r > 0$ so that for every $z \in E$, $\overline{D(z,r)} \subseteq U$. Fix such an $r > 0$. Then

$$E \subseteq E_r = \overline{\bigcup_{z \in E} D(z,r)} \subseteq U,$$

and E_r is compact. For each $z \in E$, observe that via **Cauchy Estimates 1.11.1**,

$$\left| \frac{\partial^k f_j}{\partial z^k}(z) - \frac{\partial^k f}{\partial z^k}(z) \right| \le \frac{k!}{r^k} \sup_{|\zeta - z| \le r} |f_j(\zeta) - f(\zeta)| \le \frac{k!}{r^k} \sup_{\zeta \in E_r} |f_j(\zeta) - f(\zeta)| .$$

Since E_r is compact, (f_j) converges to f uniformly on E_r. Thus

$$\sup_{\zeta \in E_r} |f_j(\zeta) - f(\zeta)| \to 0$$

as $j \to \infty$. This implies $\left(\frac{\partial^k f_j}{\partial z^k}\right)$ converges uniformly to $\frac{\partial^k f}{\partial z^k}$ on E_r. □

1.13 The Zeros of Holomorphic Functions

Definitions: discrete
Main Idea: They are isolated. As with a lot of subjects, we'll see more utility in the second semester.

Holomorphic functions are very restrictive as to where they can vanish.

Definition 1.13.1. A set S is said to be **discrete** if for each $s \in S$, there exists $\varepsilon > 0$ such that $S \cap D(s,\varepsilon) = \{s\}$. Such a set is said to contain isolated points.

Theorem 1.13.2. *Let $U \subseteq \mathbf{C}$ be a connected, open set, and let $f \in H(U)$. Let $Z = \{z \in U \mid f(z) = 0\}$. If there exists a point $z_0 \in Z$ and a sequence $(z_j) \subseteq Z \setminus \{z_0\}$ such that $\lim_{j\to\infty} z_j = z_0$, then $f \equiv 0$.*

As a consequence, the zero set of a non-identically-zero holomorphic function must be discrete.

Proof. We start by proving that $\frac{\partial^n f}{\partial z^n}(z_0) = 0$ for all $n \in \mathbf{W}$, if $z_0 \in U$ is a limit point of Z_f.

Assume to the contrary otherwise. Then there exists a smallest $n_0 \in \mathbf{Z}$ such that $\frac{\partial^{n_0} f}{\partial z^{n_0}}(z_0) \neq 0$. Then there exists an $r > 0$ so that $\overline{D(z_0,r)} \subseteq U$ and

$$f(z) = \sum_{j=n_0}^{\infty} \frac{1}{j!} \frac{\partial^j f}{\partial z^j}(z_0)(z - z_0)^j.$$

Define the function g on $\overline{D(z_0,r)}$ by

$$g(z) = \sum_{j=n_0}^{\infty} \frac{1}{j!} \frac{\partial^j f}{\partial z^j}(z_0)(z - z_0)^{j-n_0}.$$

Then $g \in H(D(z_0,r))$. Moreover, by the choice of n_0, $g(z_0) \neq 0$. However, since $g(z)(z-z_0)^{n_0} = f(z)$, $f(z_k) = 0$ for all $k \in \mathbf{N}$, and $z_k \neq z_0$, it follows that $g(z_k) = 0$ for all k. By continuity, $g(z_0) = 0$. This contradiction proves the claim; if $z_0 \in U$ is a limit point of Z_f, $\frac{\partial^n f}{\partial z^n}(z_0) = 0$ for all $n \in \mathbf{W}$. So there exists $r > 0$ so that $f \equiv 0$ on $D(z_0,r)$.

Next, let $E = \{z \in U \mid \frac{\partial^j f}{\partial z^j}(z) = 0 \text{ for all } j \in \mathbf{W}\}$. We just showed that $E \neq \emptyset$. Let's show it's clopen.

E is open, for if $w \in E$ and $r > 0$ is such that $\overline{D(w,r)} \subseteq U$, then for any $\xi \in D(w,r)$,

$$f(\xi) = \sum_{j=)}^{\infty} \frac{1}{j!} \frac{\partial^j f}{\partial w^j}(w)(\xi - w)^j \equiv 0.$$

E is closed, since

$$E = \bigcap_{j=0}^{\infty} \left(\frac{\partial^j f}{\partial z^j} \right)^{-1} (\{0\}).$$

Since E is open, closed, and nonempty, $E = U$, since U is connected. $\square$

Example 1.13.3. **Theorem 1.13.2** says that $f(z) = \sin\left(\frac{1}{z}\right)$ is not holomorphic in $D(0,1)$, while $g(z) = \sin\left(\frac{1}{z-1}\right)$ is holomorphic in $D(0,1)$.

Corollary 1.13.4. *Let $U \subseteq \mathbf{C}$ be a connected, open set, let $D(z,r) \subseteq U$, and let $f \in H(U)$. If $f|_{D(z,r)} \equiv 0$, then $f \equiv 0$ on U.*

Corollary 1.13.5. *Let $U \subseteq \mathbf{C}$ be a connected, open set. Let $f, g \in H(U)$ so that $\{z \in U \mid f(z) = g(z)\}$ has an accumulation point in U. Then $f \equiv g$.*

Corollary 1.13.6. *Let $U \subseteq \mathbf{C}$ be a connected, open set. Let $f, g \in H(U)$ satisfy $fg = 0$. Then either $f \equiv 0$ or $g \equiv 0$ on U.*

Proof. If $z_0 \in U$ and $f(z_0) \neq 0$, then by continuity, there exists $r > 0$ so that $D(z_0, r) \subseteq U$ and $f(z) \neq 0$ for any $z \in D(z_0, r)$. This means $g \equiv 0$ on $D(z_0, r)$, and hence on U by **Corollary 1.13.4**. $\square$

Corollary 1.13.7. *Let $U \subseteq \mathbf{C}$ be a connected, open set, and let $f \in H(U)$. If there exists $z_0 \in U$ so that $\frac{\partial^j f}{\partial z^j}(z_0) = 0$ for $j \in \mathbf{W}$, then $f \equiv 0$.*

Corollary 1.13.8. *If f and g are entire and $f(x) = g(x)$ for all real x, then $f \equiv g$ on $\mathbf{C}$.*

Proof. Every point of $\mathbf{R}$ is an accumulation point of $\mathbf{C}$. $\square$

Note: **Corollary 1.13.8** means that functions such as e^x, $\cos x$, $\sin x$, etc., have unique holomorphic extensions.

Also note: **Corollary 1.13.8** proves that functional identities remain true; e.g., $\cos^2 x + \sin^2 x = 1$.

Finally, we can now rigorously show that $e^z = e^x(\cos y + i \sin y) = e^x e^{iy} = e^{x+iy}$. The idea is to show that $e^z e^w = e^{z+w}$; observe that $g(w) = e^z e^w - e^{z+w}$ is entire in w for each fixed z. If $z \in \mathbf{R}$, then $g \equiv 0$ on $\mathbf{R}$. Hence $\widetilde{g}(z) = e^z e^w - e^{z+w} \equiv 0$ if $z \in \mathbf{R}$ for any $w \in \mathbf{C}$. So $\widetilde{g} \equiv 0$ for all $z \in \mathbf{C}$.

Having finally defined $\exp : \mathbf{C} \to \mathbf{C}$, it is the perfect segue into:

1.14 The Logarithm

Definitions: logarithm
Main Idea: We need to define a branch cut so that e^z is bijective and its inverse is well-defined. We discuss how to do so.

Our goal is to define the logarithm so that it agrees with the usual logarithm on $(0, \infty)$.

In $(0, \infty)$, $e^{\log x} = x$. In $\mathbf{C}$, $e^{z + i2\pi k} = e^z e^{i2\pi k} = e^z$. So e^z is periodic in $\operatorname{Im} z$. Thus, we must be careful defining the logarithm.

Lemma 1.14.1. *Let $A_{y_0} = \{z = x + iy \mid x \in \mathbf{R}, y_0 \leq y < y_0 + 2\pi\}$. Then e^z maps A_{y_0} in a one-to-one and onto manner onto $\mathbf{C} \setminus \{0\}$.*

Proof. If $e^{z_1} = e^{z_2}$, then $e^{z_1 - z_2} = 1$, so $e^{x_1 - x_2} = 1$, and $y_1 - y_2 = 2\pi k$ for some $k \in \mathbf{Z}$. But $e^{x_1} = e^{x_2}$, so $z_1 - z_2 = 2\pi i k$ for $k \in \mathbf{Z}$. If $z_1, z_2 \in A_{y_0}$, then $|y_1 - y_2| < 2\pi$. Thus, $x_1 = x_2$ and $y_1 = y_2$. Therefore, we have injectivity.

Next, let $w \in \mathbf{C} \setminus \{0\}$. We claim $e^z = w$ has a solution in A_{y_0}. We can write w as $|w| \cdot \frac{w}{|w|}$. Since $|w| \in (0, \infty)$ and $\frac{w}{|w|} \in \partial D(0,1)$, there exists $y \in [0, 2\pi)$ so that $w = e^{\log|w|} e^{iy}$. Thus, $x = \log|w|$, and by adding an appropriate multiple of 2π, say $2\pi n$, we have $x + iy + i2\pi n \in A_{y_0}$. Thus $e^z|_{A_{y_0}}$ is surjective too. $\square$

Thus from this **Lemma 1.14.1**, there is an explicit inverse for e^z on A_{y_0}.

Definition 1.14.2. The function $\log : \mathbf{C} \setminus \{0\} \to \mathbf{C}$ with range A_{y_0} is defined by $\log z = \log|z| + i \arg z$, where $\arg z \in [y_0, y_0 + 2\pi)$ and $\log|z|$ is the usual logarithm of the positive number $|z|$. We reserve $\operatorname{Log} z = \log|z| + i \arg z$ for $\arg z \in [-\pi, \pi)$. The function $\log z$ is referred to as the **branch of the logarithm** lying in A_{y_0}. The branch $A_{-\pi}$ is called the standard, or principal, branch.

Lemma 1.14.3. *For any branch of* $\log z$, $e^{\log z} = z$, *and if we choose the branch lying in* $y_0 - \pi <\leq y < y_0 + \pi$, $\log(e^z) = z$ *for any* $z = x + iy$, $y_0 - \pi \leq y < y_0 + \pi$.

Proof. Since $\log z = \log|z| + i \arg z$, $e^{\log z} = e^{\log|z|} e^{i \arg z} = |z| e^{i \arg z} = z$.

Conversely, suppose $z = x + iy$ and $y \in [y_0 - \pi, y_0 + \pi)$. By definition, $\log(e^z) = \log|e^z| + i \arg(e^z)$. See that $|e^z| = \left|e^{x+iy}\right| = e^x$, so $\log|e^z| = x$. Also, by our choice of branch, $\arg(e^z) = y$, since $y \in [y_0 - \pi, y_0 + \pi)$. $\square$

Lemma 1.14.4. *If* $z_1, z_2 \in \mathbf{C} \setminus \{0\}$, *then* $\log(z_1 z_2) = \log z_1 + \log z_2$, *up to the addition of integral multiples of* $2\pi i$.

Proof. A computation; for some $n \in \mathbf{Z}$,

$$\log(z_1 z_2) = \log|z_1 z_2| + i \arg(z_1 z_2) = \log|z_1| + \log|z_2| + i \arg z_1 + i \arg z_2 + 2\pi i n = \log z_1 + \log z_2 + 2\pi i n.$$

$\square$

Example 1.14.5. With the branch $[0, 2\pi)$, $\log((-1-i)(1-i)) = \log(-2)$. The right hand side is $\log 2 + i\pi$, whicle the left hand side is $\log((-1-i)(1-i)) = \log(-1-i) + \log(1-i) = \log(\sqrt{2}) + i\frac{5\pi}{4} + \log(\sqrt{2}) + i\frac{3\pi}{4} = \log 2 + i3\pi$.

1.15 Complex Powers

Definitions: complex power

Main Idea: Continuing with the work developed from logarithms, we discuss what it means to raise a number to a complex power. We also explore the inverse function theorem; if your derivative is nonvanishing, then your function has a local inverse.

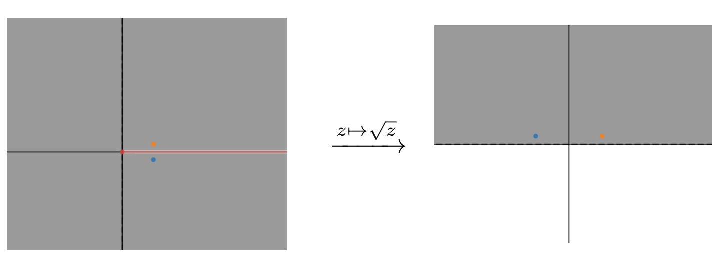
Now, our goal is to understand the value(s) for a^b for $a, b \in \mathbf{C}$. The idea here is that $a^b = \left(e^{\log a}\right)^b = e^{b \log a}$. Introducing the logarithm means we need to bear in mind branch cuts from the previous section.

Definition 1.15.1. Let $a, b \in \mathbf{C}$, $a \neq 0$. Fix a branch $A_{y_0} = \{x + iy \mid x \in \mathbf{R}, y_0 \leq y < y + 2\pi\}$. Then we define the **complex power** $a^b = e^{b \log a}$.

Proposition 1.15.2. *Let* $a, b \in \mathbf{C}$ *and* $a \neq 0$. *Then* a^b *is single-valued if and only if* $b \in \mathbf{Z}$. *If* $b \in \mathbf{Q}$ *and* $b = \frac{p}{q}$ *in lowest terms, then* a^b *has exactly* q *distinct values. If* $b \in \mathbf{R}$ *and irrational, or if* $\operatorname{Im} b \neq 0$, *then* a^b *has infinitely many values that differ by factors of the form* $e^{2\pi n b i}$ *where* $n \in \mathbf{Z}$.

The proof idea hinges on the fact that two different branches of log differ by $(2\pi i)^k$, $k \in \mathbf{Z}$.

Now let's explore branch cuts for the square root function:
If $z = re^{i\theta}$, then one square root is $\sqrt{z} = \sqrt{r} e^{i\frac{\theta}{2}}$. Note that square root cannot be continuous across the branch, though it can be defined everywhere! The following picture demonstrates; though the blue and orange points are near each other, they are not in the image:



For holomorphicity of $\sqrt{z}$, we exclude the branch cut.

Recall the real variables inversion function theorem, which we will need for the proof of a complex analog:

Theorem 1.15.3. *If $\Omega \subseteq \mathbf{R}^2$ and $f \in C^1(\Omega)$, where $Df(x_0, y_0)$ is nonsingular, then there are neighborhoods U of (x_0, y_0) and V of $f(x_0, y_0)$ such that $f : U \to V$ is a bijection and $f^{-1} : V \to U$ is differentiable, with $Df^{-1}(f(x,y)) = (Df(x,y))^{-1}$.*

Theorem 1.15.4 (Inverse Function Theorem). *Let $f \in H(\Omega)$ and assume $f'(z_0) \neq 0$. Then there exists a neighborhood U of z_0 and a neighborhood V of $f(z_0)$ such that $f : U \to V$ is a bijection, and its inverse function f^{-1} is holomorphic on V with complex derivative $\frac{d}{dw} f^{-1}(w) = \frac{1}{f'(z)}$, where $w = f(z)$.*

Note that the inverse is a local property.

Proof. Let $f = u + iv \in H(\Omega)$. Then

$$Df = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u}{\partial x} & -\dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial u}{\partial x} \end{bmatrix}.$$

Consequently, $|Df| = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 = |f'(z)|^2$. Thus, if $z_0 = x_0 + iy_0$, $f'(z_0) \neq 0$ means that $|Df(x_0, y_0)| = |f'(z_0)|^2 \neq 0$. Thus, the real inverse function theorem applies.

Now, since

$$Df = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix},$$

we get that

$$(Df)^{-1} = \frac{1}{\det Df} \begin{bmatrix} \dfrac{\partial v}{\partial y} & -\dfrac{\partial u}{\partial y} \\ -\dfrac{\partial v}{\partial x} & \dfrac{\partial u}{\partial x} \end{bmatrix}.$$

Writing $f^{-1}(x,y) = t(x,y) + is(x,y)$, it follows that

$$D\left(f^{-1}\right) = \begin{bmatrix} \dfrac{\partial t}{\partial x} & \dfrac{\partial t}{\partial y} \\ \dfrac{\partial s}{\partial x} & \dfrac{\partial s}{\partial y} \end{bmatrix}.$$

By the real inverse function theorem,

$$\begin{aligned} \frac{\partial t}{\partial x} &= \frac{1}{\det Df} \frac{\partial v}{\partial y} = \frac{1}{\det Df} \frac{\partial u}{\partial x}, \\ \frac{\partial s}{\partial x} &= \frac{1}{\det Df} \left(-\frac{\partial v}{\partial x}\right) = \frac{1}{\det Df} \frac{\partial u}{\partial y}, \end{aligned}$$

of course evaluated at $f(x,y)$. Similarly,

$$\begin{aligned} \frac{\partial t}{\partial y} &= \frac{1}{\det Df} \frac{\partial v}{\partial x}, \text{ and} \\ \frac{\partial s}{\partial y} &= \frac{1}{\det Df} \frac{\partial v}{\partial y}. \end{aligned}$$

Thus, the **Cauchy-Riemann 1.5.4** equations hold for t and s. It follows that $f^{-1} \in H(V)$, and that

$$\left(f^{-1}\right)' = \frac{dt}{dx} + i\frac{ds}{dx} = \frac{1}{\det Df}\left(\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}\right) = \frac{\overline{f'(z)}}{|f'(z)|^2} = \frac{1}{f'(z)}.$$

□

A comment to make here: $\log \in H(\mathrm{Int}(A_{y_0}))$, where $\mathrm{Int}(A_{y_0}) = \{z = x + iy \mid y_0 < y < y_0 + 2\pi\}$, and $\frac{\partial}{\partial z}\log z = \frac{1}{z}$, since, much like logarithmic differentiation in calculus,

$$1 = \frac{\partial}{\partial z}[z] = \frac{\partial}{\partial z}\left[e^{\log z}\right] = e^{\log z}\frac{\partial}{\partial z}[\log z] = z\frac{\partial}{\partial z}[\log z].$$

1.16 Meromorphic Functions and Residues

Definitions: isolated singularity, Laurent series, annulus, pole of order k, principal part, holomorphically simply connected, index, meromorphic, singularity at ∞

Main Idea: We begin with a discussion of the types of singularities possible. Removable are bounded nearby, poles go to ∞, and essential singularties are everything else. If a singularity is removable, there is a holomorphic extension across it. Laurent series are a generalization of power series to negative exponents, and we can use a Laurent series expansion about a singularity to see whether the singularity is removable, a pole, or essential. We also get to the Residue Theorem, a powerful tool in computing contour integrals. We use the theorem to compute several tricky real valued integrals and sums, using clever choices of complex extensions and contours. Finally, we define meromorphic functions (functions with only poles), and talk about the behavior of singularties at ∞.

In this section, we will study the behavior near a singularity. We have already seen one important example: the **Cauchy Integral Formula 1.9.3**.

Definition 1.16.1. Let $U \subseteq \mathbf{C}$ be open and $z_0 \in U$. Suppose $f \in H(U \setminus \{z_0\})$. Then we say that f has an **isolated singularity** at z_0, or equivalently, that z_0 is an isolated singularity of f.

There are three possible cases of singularities:

1. For some $r > 0$, $|f(z)|$ is bounded on $D(z_0, r) \setminus \{z_0\}$. This means that there exists $M > 0$ so that $|f(z)| \le M$. This is called a **removable singularity**.
2. $\lim_{z \to z_0} |f(z)| = \infty$. This is called a **pole**.
3. Neither 1. nor 2. applies. This is called an **essential singularity**.

Theorem 1.16.2 (Riemann Removable Singularities Theorem). *Let $f \in H(D(z_0, r) \setminus \{z_0\})$ be bounded. Then*

1. *$\lim_{z \to z_0} f(z)$ exists, and*
2. *the function $\widehat{f} : D(z_0, r) \to \mathbf{C}$ defined by*

$$\widehat{f}(z) = \begin{cases} f(z) & \text{if } z \in D(z_0, r) \setminus \{z_0\}; \\ \lim_{\zeta \to z_0} f(\zeta) & \text{if } z = z_0 \end{cases}$$

is holomorphic on $D(z_0, r)$.

Proof. Set $g : D(z_0, r) \to \mathbf{C}$ to be

$$g(z) = \begin{cases} (z - z_0)^2 f(z) & \text{if } z \in D(z_0, r) \setminus \{z_0\}; \\ 0 & \text{if } z = z_0. \end{cases}$$

We will show that $g \in H(D(z_0, r))$, and conclude from it that $\widehat{f} \in H(D(z_0, r))$.

We first claim that $g \in C^1(D(z_0, r))$. This is clear away from z_0, so to see that $g \in C^1$, we first show that g is differentiable at z_0. Using the fact that f is bounded, we see that

$$|g(z_0) - g(z)| = |z - z_0|^2 |f(z)| \leq M|z - z_0|^2 \to 0$$

as $z \to z_0$. Thus $g \in C(D(z_0, r))$. Next,

$$\left|\frac{g(z_0 + h) - g(z_0)}{h}\right| = \left|\frac{(z_0 + h - z_0)^2}{h} f(z_0 + h)\right| = |h||f(z_0 + h)| \leq |h|M \to 0$$

as $h \to 0$. By taking h real, we have shown $\frac{\partial g}{\partial x}(z_0) = 0$, and h purely imaginary, we have shown $\frac{\partial g}{\partial y}(z_0) = 0$. We now show that the partial derivatives of g are continuous at z_0; namely,

$$\lim_{z \to z_0} \frac{\partial g}{\partial x}(z) = \lim_{z \to z_0} \frac{\partial g}{\partial y}(z) = 0.$$

We show the first, using **Cauchy Estimates 1.11.1**. Since f is bounded by M, if $w \in D\left(z_0, \frac{r}{2}\right) \setminus \{z_0\}$, then

$$\left|\frac{\partial f}{\partial z}(w)\right| \leq \frac{M}{|w - z_0|}.$$

Since f is holomorphic near w, $\frac{\partial f}{\partial z}(w) = \frac{\partial f}{\partial x}(w)$. Thus

$$\left|\frac{\partial f}{\partial x}(w)\right| \leq \frac{M}{|w - z_0|}.$$

Since g is holomorphic near w,

$$\left|\frac{\partial g}{\partial x}(w)\right| = \left|\frac{\partial g}{\partial z}(w)\right| = \left|2|w - z_0|f(w) + |w - z_0|^2 f'(w)\right| \leq 2|w - z_0|M + |w - z_0|M \to 0$$

as $w \to z_0$. Therefore, $g \in C^1(D(z_0, r))$, as we wished to show.

Next, we need to show that g, and more importantly, $\widehat{f}$, are holomorphic. First, to see $g \in H(D(z_0, r))$, see that

$$\frac{\partial g}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}}\left[(z - z_0)^2 f(z)\right] = 0;$$

thus $g \in H(D(z_0, r) \setminus \{z_0\})$, and since $g \in C^1$, by **Theorem 1.7.4**, $g \in H(D(z_0, r))$.

Finally, to see $\widehat{f}$ is holomorphic, note that since $g(z_0) = g'(z_0) = 0$, the Taylor series expansion for g has a radius of convergence of at least r, and

$$g(z) = \sum_{k=2}^{\infty} a_k (z - z_0)^k = (z - z_0)^2 \sum_{k=2}^{\infty} a_k (z - z_0)^{k-2} = (z - z_0)^2 H(z).$$

The holomorphic function $H(z)$ also has a radius of convergence of at least r, and therefore $H \in H(D(z_0, r))$ and satisfies $g(z) = (z - z_0)^2 H(z)$. Therefore $\widehat{f} = H$ is the desired extension. $\square$

Warning: not every singularity is removable!

Example 1.16.3. Let $f(z) = \exp\left(\frac{1}{z}\right)$. Then $f \in H(D(0, 1) \setminus \{0\})$.

Let $\varepsilon > 0$ and let $\alpha \in \mathbf{C} \setminus \{0\}$. We claim that there exists $z \in D(0, \varepsilon) \setminus \{0\}$ with $\exp\left(\frac{1}{z}\right) = \alpha$. (Good *God*, this is as far from removable as you can get.)

To see this, first note that certainly, there exists $w \in \mathbf{C} \setminus \{0\}$ so that $e^w = \alpha$. By the periodicity of e^z, we know that $e^{w + i2\pi k} = \alpha$ as well, for all $k \in \mathbf{Z}$. Therefore, if $z_k = \frac{1}{w + 2\pi i k}$, then $\exp\left(\frac{1}{z_k}\right) = \alpha$, and $z_k \to 0$.

How bad can the behavior really be?

Theorem 1.16.4 (Casorati-Weierstrass Theorem). *If $f \in H(D(z_0, r_0) \setminus \{z_0\})$ and z_0 is an essential singularity of f, then $f(D(z_0, r) \setminus \{z_0\})$ is dense in $\mathbf{C}$ for every $0 < r < r_0$.*

(And this isn't even the worst of it; not only is the image dense, it turns out that it can miss at most one point in $\mathbf{C}$.[4] Good ***God***.)

Proof. Suppose the statement of the theorem does not hold for some $0 < r < r_0$. Then, there exists $\lambda \in \mathbf{C}$ and $\varepsilon > 0$ such that $|f(z) - \lambda| > \varepsilon$ for all $z \in D(z_0, r) \setminus \{z_0\}$. Define $g \in H(D(z_0, r) \setminus \{z_0\})$ by $g(z) = \frac{1}{f(z)-\lambda}$. Then $|g(z)| < \frac{1}{\varepsilon}$ for $z \in D(z_0, r) \setminus \{z_0\}$. By the **Riemann Removable Singularities Theorem 1.16.2**, there exists $\widehat{g} \in H(D(z_0, r))$ such that $\widehat{g}(z) = g(z)$ for all $z \in D(z_0, r) \setminus \{z_0\}$. There are now two possibilites; at z_0, we can either have:

1. $\widehat{g}(z_0) \neq 0$. Then f has a removable singularity at z_0.
2. $\widehat{g}(z_0) = 0$. This forces $|f(z)| \to \infty$ as $z \to z_0$; i.e., f has a pole at z_0.

In either case, f would not have an essential singularity at z_0, a contradiction. $\square$

Now we are going to develop a "power series" expansion for functions that have a pole or an essential singularity. Fortunately, all that is required are negative exponents!

Definition 1.16.5. A **Laurent series** on $D(z_0, r)$ is a (formal) expression of the form

$$\sum_{j=-\infty}^{\infty} a_j (z - z_0)^j.$$

A few comments:

1. The terms $a_j(z - z_0)^j$ are defined on $D(z_0, r) \setminus \{z_0\}$ for $j < 0$.
2. We say that a series

$$\sum_{j=-\infty}^{\infty} \alpha_j$$

converges if and only if both

$$\sum_{j=0}^{\infty} \alpha_j \qquad \text{and} \qquad \sum_{j=1}^{\infty} \alpha_{-j}$$

converge. In terms of the formalism, this means that

$$\sum_{j=-\infty}^{\infty} \alpha_j = \sigma$$

if for all $\varepsilon > 0$ there exists $N = N(\varepsilon) > 0$ so that if $k, \ell \geq N$, then

$$\left| \sum_{j=-k}^{\ell} \alpha_j - \sigma \right| < \varepsilon.$$

[4]This is Picard's Great Theorem, not proven in this class. We didn't do anything Picard, unfortunately.

Warning: based on 2. above,

$$\sum_{j \neq 0} \frac{1}{j}$$

diverges, but

$$\lim_{n \to \infty} \sum_{j=-n}^{n} \frac{1}{j} = 0.$$

Don't ever sum in this way; only consider the convergence/divergence of the positively indexed terms and the negatively indexed terms on their own one at a time.

Lemma 1.16.6. *If the Laurent series*

$$\sum_{j=-\infty}^{\infty} a_j (z - z_0)^j$$

converges at $z_1, z_2 \notin \{z_0\}$*, and* $|z_1 - z_0| < |z_2 - z_0|$*, then if* $|z_1 - z_0| < |z - z_0| < |z_2 - z_0|$*, then the series converges on that annulus.*

Proof. Since

$$\sum_{j=-\infty}^{\infty} a_j (z_2 - z_0)^j$$

converges, then by definition, so does

$$\sum_{j=0}^{\infty} a_j (z_2 - z_0)^j.$$

Then, by **Abel's Lemma 1.10.6**,

$$\sum_{j=0}^{\infty} a_j (z - z_0)^j$$

converges when $|z - z_0| < |z_2 - z_0|$.

Similarly, the series

$$\sum_{j=-\infty}^{-1} a_j (z_1 - z_0)^j = \sum_{j=1}^{\infty} a_{-j} (z_1 - z_0)^{-j}$$

converges as well. Since $\frac{1}{|z - z_0|} < \frac{1}{|z_1 - z_0|}$, **Abel's Lemma 1.10.6** again implies that

$$\sum_{j=1}^{\infty} a_{-j} (z - z_0)^{-j}$$

converges. □

From **Lemma 1.16.6**, it is apparent that a Laurent series will converge on a set which is of the form $\{z \mid 0 \leq r_1 < |z - z_0| < r_2\}$, together with some or all of the points satisfying $|z - z_0| = r_1$ or $|z - z_0| = r_2$.

Definition 1.16.7. An open set of the form $\{z \mid 0 \leq r_1 < |z - z_0| < r_2\}$ or $\{z \mid 0 \leq r_1 < |z - z_0| < \infty\}$ is called an **annulus** centered at z_0.

Annuli are of the form $D(z_0, r_2) \setminus \overline{D(z_0, r_1)}$.

Thus, we can restate **Lemma 1.16.6**:

Lemma 1.16.8. *Let*

$$\sum_{j=-\infty}^{\infty} a_j(z-z_0)^j$$

be a doubly infinite series that converges at at least one point. Then there are unique, nonnegative numbers r_1 *and* r_2 *(with* r_2 *possibily infinite) such that the series converges absolutely for all* z *such that* $r_1 < |z-z_0| < r_2$ *and diverges for* $|z-z_0| < r$ *or* $|z-z_0| > r_2$*.*

Also, if $r_1 < {r_1}' \leq {r_2}' < r_2$*, then*

$$\sum_{j=-\infty}^{\infty} \left| a_j(z-z_0)^j \right|$$

converges uniformly on $\{z \mid {r_1}' \leq |z-z_0 \leq {r_2}'\}$*, and consequently*

$$\sum_{j=-\infty}^{\infty} a_j(z-z_0)^j$$

converges absolutely and uniformly there.

Example 1.16.9. We compute a Laurent expansion for $\frac{z+1}{z}$ around $z_0 = 0$. It is $1 + \frac{1}{z}$.

Example 1.16.10. We compute a Laurent expansion for $\frac{z}{1+z^2}$ around $z_0 = i$. See that

$$\frac{z}{1+z^2} = \frac{z}{(z+i)(z-i)} = \frac{1}{2} \cdot \frac{1}{z-i} + \frac{1}{2} \cdot \frac{1}{z+i},$$

via partial fractions. Now, $\frac{1}{2} \cdot \frac{1}{z-i}$ is good, while $\frac{1}{2} \cdot \frac{1}{z+i}$ is not, as it's not centered at i. To fix this,

$$\frac{1}{z+i} = \frac{1}{2i+z-i} = \frac{1}{2i} \cdot \frac{1}{1-\left(-\frac{z-i}{2i}\right)} = \frac{1}{2i}\sum_{j=0}^{\infty}(-1)^j \left(\frac{z-i}{2i}\right)^j = \sum_{j=0}^{\infty} i^{j-1}2^{-j-1}(z-i)^j,$$

for $|z-i| < 2$. Therefore,

$$\frac{z}{1+z^2} = \frac{1}{2}(z-i)^{-1} + \sum_{j=0}^{\infty} i^{j-1}2^{-j-2}(z-i)^j,$$

which is convergent on $\{z \mid 0 < |z-i| < 2\}$.

Lemma 1.16.11. *Let* $0 \leq r_1 < r_2 < \infty$*. If the Laurent series*

$$\sum_{j=-\infty}^{\infty} a_j(z-z_0)^j$$

converges on $D(z_0, r_2) \setminus \overline{D(z_0, r_1)}$ *to a function* f*, then for any* r *satisfying* $r_1 < r < r_2$ *and each* $j \in \mathbf{Z}$*,*

$$a_j = \frac{1}{2\pi i} \oint_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} d\zeta.$$

In particular, a_j *is uniquely determined by* f*.*

Proof. By **Lemma 1.16.8**, the series converges absolutely and uniformly in ζ on $\{\zeta \mid |\zeta - z_0| = r\}$. Therefore, we may interchange the sum and integral as follows:

$$\oint_{|\zeta-z_0|=r} \frac{f(\zeta)}{(\zeta-z_0)^{j+1}} d\zeta = \oint_{|\zeta-z_0|=r} \sum_{k=-\infty}^{\infty} a_k(\zeta-z_0)^{k-j-1} d\zeta = \sum_{k=-\infty}^{\infty} a_k \oint_{|\zeta-z_0|=r} (\zeta-z_0)^{k-j-1} d\zeta = a_j 2\pi i.$$

□

Theorem 1.16.12 (**The Cauchy Integral Formula for an Annulus**). *Suppose that* $0 \le r_1 < r_2 \le \infty$ *and* $f \in H(D(z_0, r_2) \setminus \overline{D(z_0, r_1)})$. *Then for each* s_1 *and* s_2 *so that* $r_1 < s_1 < s_2 < r_2$ *and for each* $z \in D(z_0, s_2) \setminus \overline{D(z_0, s_1)}$, *it holds that*

$$f(z) = \frac{1}{2\pi i} \oint_{|\zeta - z_0| = s_2} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \oint_{|\zeta - z_0| = s_1} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Proof. This follows immediately from the **Generalized/Inhomogeneous Cauchy Integral Formula 1.9.6**. ☐

Theorem 1.16.13. *If* $0 \le r_1 < r_2 \le \infty$ *and* $f : D(z_0, r_2) \setminus \overline{D(z_0, r_1)} \to \mathbf{C}$ *is holomorphic, then there exists* $(a_j) \subseteq \mathbf{C}$ *such that*

$$\sum_{j=-\infty}^{\infty} a_j (z - z_0)^j$$

converges on $D(z_0, r_2) \setminus \overline{D(z_0, r_1)}$ *to* f. *If* $r_1 < s_1 < s_2 < r_2$, *then the series converges absolutely and uniformly on* $\overline{D(z_0, s_2)} \setminus D(z_0, s_1)$.

This theorem guarantees the existence of Laurent expansions.

Proof. Suppose z satisfies $0 \le r_1 < s_1 < |z - z_0| < s_2 < r_2$. Then the two integrals in the expression

$$f(z) = \frac{1}{2\pi i} \oint_{|\zeta - z_0| = s_2} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \oint_{|\zeta - z_0| = s_1} \frac{f(\zeta)}{\zeta - z} d\zeta$$

can be expanded in series. In particular, we have

$$\begin{aligned}
\oint_{|\zeta - z_0| = s_2} \frac{f(\zeta)}{\zeta - z} d\zeta &= \oint_{|\zeta - z_0| = s_2} \frac{f(\zeta)}{1 - \frac{z - z_0}{\zeta - z_0}} \cdot \frac{1}{\zeta - z_0} d\zeta \\
&= \oint_{|\zeta - z_0| = s_2} \frac{f(\zeta)}{\zeta - z_0} \sum_{j=0}^{\infty} \frac{(z - z_0)^j}{(\zeta - z_0)^j} d\zeta \\
&= \oint_{|\zeta - z_0| = s_2} f(\zeta) \sum_{j=0}^{\infty} \frac{(z - z_0)^j}{(\zeta - z_0)^{j+1}} d\zeta.
\end{aligned}$$

The geometric series for $\frac{1}{1 - \frac{z - z_0}{\zeta - z_0}}$ converges absolutely and uniformly in $\zeta \in \partial D(z_0, s_2)$, since $\frac{|z - z_0|}{|\zeta - z_0|} = \frac{|z - z_0|}{s_2} < 1$. This means that we can switch the order of summation, and compute

$$\oint_{|\zeta - z_0| = s_2} \frac{f(\zeta)}{\zeta - z} d\zeta = \sum_{j=0}^{\infty} \left(\oint_{|\zeta - z_0| = s_2} \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} d\zeta \right) (z - z_0)^j.$$

Similarly, for $s_1 < |z - z_0|$,

$$\begin{aligned}
\oint_{|\zeta - z_0| = s_1} \frac{f(\zeta)}{\zeta - z} d\zeta &= - \oint_{|\zeta - z_0| = s_1} \frac{f(\zeta)}{1 - \frac{\zeta - z_0}{z - z_0}} \cdot \frac{1}{z - z_0} d\zeta \\
&= - \oint_{|\zeta - s_0| = s_1} \frac{f(\zeta)}{z - z_0} \sum_{j=0}^{\infty} \frac{(\zeta - z_0)^j}{(z - z_0)^j} d\zeta \\
&= - \sum_{j=0}^{\infty} \left(\oint_{|\zeta - z_0| = s_1} f(\zeta)(\zeta - z_0)^j d\zeta \right) (z - z_0)^{-j-1} \\
&= - \sum_{j=-\infty}^{-1} \left(\oint_{|\zeta - z_0| = s_1} f(\zeta)(\zeta - z_0)^{-(j+1)} d\zeta \right) (z - z_0)^j.
\end{aligned}$$

Thus,

$$2\pi i f(z) = \sum_{j=-\infty}^{-1} \left(\oint_{|\zeta - z_0|=s_1} \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} d\zeta \right) (z - z_0)^j + \sum_{j=0}^{\infty} \left(\oint_{|\zeta - z_0|=s_2} \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} d\zeta \right) (z - z_0)^j.$$

The series converges absolutely and uniformly as a consequence of **Lemma 1.16.8**. $\square$

Note that the series derived in the proof of **Theorem 1.16.13**,

$$2\pi i f(z) = \sum_{j=-\infty}^{-1} \left(\oint_{|\zeta - z_0|=s_1} \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} d\zeta \right) (z - z_0)^j + \sum_{j=0}^{\infty} \left(\oint_{|\zeta - z_0|=s_2} \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} d\zeta \right) (z - z_0)^j,$$

is independent of the choice of s_1 and s_2, by the **Deformation Theorem 1.9.19**.

Lemma 1.16.14. *If $f \in H(D(z_0, r) \setminus \{z_0\})$, then f has a unique Laurent series expansion*

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z - z_0)^j$$

which converges absolutely and uniformly on compact subsets of $D(z_0, r) \setminus \{z_0\}$. The coefficients are given by

$$a_j = \frac{1}{2\pi i} \oint_{\partial D(z_0, s)} \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} d\zeta$$

for any $0 < s < r$. One of the following three cases must happen:

1. *$a_j = 0$ for all $j < 0$,*
2. *for some $k > 0$, $a_j = 0$ for all $-\infty < j \leq -k$, or*
3. *neither 1. nor 2. applies.*

These correspond exactly to the cases that

1. *z_0 is a removable singularity,*
2. *z_0 is a pole, and*
3. *z_0 is an essential singularity.*

Proof. We begin by showing that $a_j = 0$ for all $j < 0$ if and only if z_0 is a removable singularity. For one direction, the Laurent series is a power series centered at z_0 that converges on $D(z_0, r)$. Thus, the power series converges to a holomorphic function on $D(z_0, r)$ that agrees with f on $D(z_0, r) \setminus \{z_0\}$. Also, it follows immediately that $|f|$ is bounded near z_0.

For the other direction, assume z_0 is removable. We need to show that $a_j = 0$ for all $j < 0$, where

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z - z_0)^j.$$

Let $\widehat{f}$ be the holomorphic extension of f to $D(z_0, r)$. Then $\widehat{f}$ has the power series expansion

$$\widehat{f}(z) = \sum_{j=0}^{\infty} \widehat{a_j} (z - z_0)^j.$$

By the uniqueness of the Laurent series, $\widehat{a_j} = a_j$ for all j. Hence $a_j = 0$ whenever $j < 0$.

Now, we show that for some $k > 0$, $a_j = 0$ for all $-\infty < j \leq -k$ if and only if z_0 is a pole. In one direction, if $k > 0$ and

$$f(z) = \sum_{j=-k}^{\infty} a_j (z - z_0)^j$$

with $a_{-k} \neq 0$, then

$$|f(z)| \geq |z - z_0|^{-k} \left(|a_{-k}| - \left| \sum_{j=-k+1}^{\infty} a_j (z - z_0)^{j+k} \right| \right).$$

Observe that

$$\sum_{j=-k+1}^{\infty} a_j (z - z_0)^{j+k}$$

is a power series with positive radius of convergence, so

$$\lim_{z \to z_0} \left| \sum_{j=-k+1}^{\infty} a_j (z - z_0)^{j+k} \right| = 0;$$

hence,

$$|f(z)| \geq |z - z_0|^{-k} \left(|a_{-k}| - \frac{1}{2} |a_{-k}| \right)$$

if $|z - z_0|$ is sufficiently small. Thus, $\lim_{z \to z_0} |f(z)| = \infty$.

For the other direction, since $\lim_{z \to z_0} |f(z)| = \infty$, there exists $0 < s < r$ so that $f(z) \neq 0$ on $D(z_0, s) \setminus \{z_0\}$. Set $g(z) = \frac{1}{f(z)}$ on $D(z_0, s) \setminus \{z_0\}$. It follows that $\lim_{z \to z_0} g(z) = 0$. Also, g is holomorphic on $D(z_0, s) \setminus \{z_0\}$. By the **Riemann Removable Singularities Theorem 1.16.2**, the function

$$H(z) = \begin{cases} g(z) & \text{if } z \in D(z_0, s) \setminus \{z_0\}; \\ 0 & \text{if } z = z_0 \end{cases}$$

is holomorphic on $D(z_0, s)$. Since $H \not\equiv 0$, there exists $m \geq 1$ so that

$$H(z) = \sum_{j=m}^{\infty} a_j (z - z_0)^j = (z - z_0)^m \sum_{j=m}^{\infty} a_j (z - z_0)^{j-m},$$

where $a_m \neq 0$. It follows that

$$Q(z) = \sum_{j=m}^{\infty} a_j (z - z_0)^{j-m}$$

is nonvanishing on $D(z_0, s)$. Thus, $\frac{1}{Q(z)}$ is holomorphic on $D(z_0, s)$, and on $D(z_0, s) \setminus \{z_0\}$,

$$f(z) = \frac{1}{H(z)} = (z - z_0)^{-m} \frac{1}{Q(z)} = (z - z_0)^{-m} \sum_{j=0}^{\infty} b_j (z - z_0)^j = \sum_{k=-m}^{\infty} b_{k+m} (z - z_0)^k.$$

By the uniqueness of Laurent series expansions, this series is the expansion of f on $D(z_0, s) \setminus \{z_0\}$, so it is shown.

To see the characterization of an essential singularity in a Laurent series is trivial; both third cases are the result of "neither 1. nor 2. apply." □

Definition 1.16.15. If f has a Laurent expansion

$$f(z) = \sum_{j=-k}^{\infty} a_j (z - z_0)^j$$

for some $k > 0$ and $a_{-k} \neq 0$, then we say that f has a **pole of order** k at z_0.

Note that f has a pole of order k if and only if $(z - z_0)^k f(z)$ is bounded near z_0 and $(z - z_0)^{k-1} f(z)$ is not.

Definition 1.16.16. If

$$f(z) = \sum_{j=-k}^{\infty} a_j (z - z_0)^j$$

has a pole of order k at z_0, we call

$$\sum_{j=-k}^{-1} a_j (z - z_0)^j$$

the **principal part** of f at z_0.

Proposition 1.16.17. *Let $f \in H(D(z_0, r) \setminus \{z_0\})$ and suppose f has a pole of order k at z_0. Then the Laurent series coefficients a_j of f about z_0 are given by*

$$a_j = \frac{1}{(k+j)!} \frac{\partial^{k+j}}{\partial z^{k+j}} \left[(z - z_0)^k f(z) \right] \Big|_{z=z_0}.$$

Example 1.16.18. Let $f(z) = \frac{z+1}{z-1}$ and center the Laurent series at $z_0 = 1$. The annulus of consideration is $D(1, 2) \setminus \{1\}$.

First recognize that

$$\frac{z+1}{z-1} = \frac{z-1+2}{z-1} = \frac{2}{z-1} + 1.$$

Then, $g(z) = (z-1)f(z) = z + 1$, $a_{-1} = g(1) = 2$, $a_0 = g'(1) = 1$, or

$$a_j = \frac{1}{2\pi i} \oint_{\partial D(1,1)} \frac{f(\zeta)}{(\zeta - 1)^{j+1}} d\zeta,$$

when $j > 0$.

Example 1.16.19. Let $f(z) = \frac{e^z}{(z-i)^2 (z-2)^3}$. Note immediately that $f \in H(\mathbf{C} \setminus \{i, 2\})$.

Let's figure out the prinicple part of f at $z_0 = 2$. We know that

$$a_j = \frac{1}{(k+j)!} \frac{\partial^{k+j}}{\partial z^{k+j}} \left[(z - z_0)^k f \right] \Big|_{z=z_0}.$$

Here, $z_0 = 2$ and $k = 3$. Therefore,

$$\begin{aligned}
a_{-3} &= \frac{1}{0!} (z-2)^3 \frac{e^z}{(z-i)^2 (z-2)^3} \Big|_{z=2} = \frac{e^2}{(2-i)^2}, \\
a_{-2} &= \frac{1}{1!} \frac{\partial}{\partial z} \left[\frac{e^z}{(z-i)^2} \right] \Big|_{z=2} = \left(e^z (z-i)^{-2} - 2e^z (z-i)^{-3} \right) \Big|_{z=2} = e^2 (2-i)^{-2} - 2e^2 (2-i)^{-3} = \frac{-ie^2}{(2-i)^3}, \\
a_{-1} &= \frac{1}{2!} \frac{\partial^2}{\partial z^2} \left[\frac{e^z}{(z-i)^2} \right] \Big|_{z=2} = \cdots \quad \text{(yuck, computation. If I'm so inclined, I may} \\
&\qquad \text{work this out once the notes are done.)}
\end{aligned}$$

We now turn our attention to residues. The goal is to study the following situation:

Let $U \subseteq \mathbf{C}$ be open, and $\{P_1, ..., P_n\} \subseteq U$. If $f \in H(U \setminus \{P_1, ..., P_n\})$ and $\gamma : [0,1] \to U \setminus \{P_1, ..., P_n\}$ is a piecewise C^1 curve, then how does the behavior of f near $\{P_1, ..., P_n\}$ affect $\oint_\gamma f$?

Definition 1.16.20. An open set $U \subseteq \mathbf{C}$ is called **holomorphically simply connected** if U is connected, and for every $f \in H(U)$, there exists $F \in H(U)$ such that $F' = f$.

Proposition 1.16.21. *A connected open set $U \subseteq \mathbf{C}$ is holomorphically simply connected if and only if U is simply connected if and only if for each $f \in H(U)$ and each piecewise C^1 closed curve γ in U, $\oint_\gamma f = 0$.*

The proof of this is heavily steeped in second semester stuff, like topology. We will get there! See the **Riemann Mapping Theorem 2.2.2** and subsequent results.

Definition 1.16.22. If $\gamma : [a,b] \to \mathbf{C}$ is a piecewise C^1 closed curve, and if $P \notin \widetilde{\gamma} = \gamma([a,b])$, then the **index of γ with respect to** P, written $\mathrm{Ind}_\gamma(P)$, is defined by

$$\mathrm{Ind}_\gamma(P) = \frac{1}{2\pi i} \oint_\gamma \frac{1}{\zeta - P} d\zeta.$$

The index is also called the **winding number**. Informally, the index returns the number of times γ loops counterclockwise around P.

Lemma 1.16.23. *If $\gamma : [a,b] \to \mathbf{C} \setminus \{P\}$ is a piecewise C^1 closed curve, then*

$$\mathrm{Ind}_\gamma(P) = \frac{1}{2\pi i} \oint_\gamma \frac{1}{\zeta - P} d\zeta = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t) - P} dt$$

is an integer.

Thus, the informal intuition of index should begin to make more sense; a closed curve wraps an integral number of times around a point.

Proof. Set

$$g(t) = (\gamma(t) - P) \exp\left(-\int_a^t \frac{\gamma'(s)}{\gamma(s) - P} ds\right).$$

The function g is piecewise C^1, because γ is.

We first will show that g is constant. To do this, it suffices to show that $g'(t) = 0$ for every t at which $\gamma'(t)$ exists. By the product rule,

$$g'(t) = \gamma'(t) \exp\left(-\int_a^t \frac{\gamma'(s)}{\gamma(s) - P} ds\right) + (\gamma(t) - P) \frac{-\gamma'(t)}{\gamma(t) - P} \exp\left(-\int_a^t \frac{\gamma'(s)}{\gamma(s) - P} ds\right) = 0.$$

Thus g is constant.

Next, we evaluate g at $t = a$ and $t = b$, and use the fact that $\gamma(a) = \gamma(b)$. See that

$$\begin{aligned} g(a) &= \gamma(a) - P, \\ g(b) &= (\gamma(b) - P) \exp\left(-\int_a^b \frac{\gamma'(s)}{\gamma(s) - P} ds\right) = (\gamma(a) - P) \exp\left(-\int_a^b \frac{\gamma'(s)}{\gamma(s) - P} ds\right). \end{aligned}$$

Since $g(a) = g(b)$,

$$\exp\left(-\int_a^b \frac{\gamma'(s)}{\gamma(s) - P} ds\right) = 1,$$

hence

$$\int_a^b \frac{\gamma'(s)}{\gamma(s) - P} ds$$

must be an integer multiple of $2\pi i$. $\square$

Theorem 1.16.24 (**The Residue Theorem**). *Suppose that $U \subseteq \mathbf{C}$ is simply connected and $P_1, ..., P_n$ are distinct points in U. Suppose $f \in H(U \setminus \{P_1, ..., P_n\})$ and γ is a closed, piecewise C^1 curve in $U \setminus \{P_1, ..., P_n\}$. Set R_j, also written* $\mathrm{Res}(f, P_j)$ *or* $\mathrm{Res}_f(P_j)$, *equal to the coefficient of $(z - P_j)^{-1}$ in the Laurent series of f about P_j. Then*

$$\oint_\gamma f = \sum_{j=1}^{n} R_j \left(\oint_\gamma \frac{1}{\zeta - P_j} d\zeta \right) = \sum_{j=1}^{n} \mathrm{Res}_f(P_j) 2\pi i \,\mathrm{Ind}_\gamma(P_j).$$

Proof. For each $j \in \{1, ..., n\}$, expand f in a Laurent series about P_j. Let s_j be the principal part of f about P_j. The series defining s_j is a convergent power series in $(z - P_j)^{-1}$, and defines a holomorphic function on $\mathbf{C} \setminus \{P_j\}$. We can therefore write $f = (f - (s_1 + \cdots + s_n)) + (s_1 + \cdots + s_n)$. Observe that $f - (s_1 + \cdots + s_n)$ and $s_1 + \cdots + s_n$ are defined on $U \setminus \{P_1, ..., P_n\}$. Also, $f - (s_1 + \cdots + s_n)$ was constructed so that the singularities, $P_1, ..., P_n$, are removable; this is because $f - s_j$ has a Laurent series with no negative exponents at P_j, and s_k is holomorphic near P_j if $j \neq k$.

Next, U is simply connected, so

$$\oint_\gamma \left(f - \sum_{j=1}^{n} s_j \right) = 0;$$

hence

$$\oint_\gamma f = \sum_{j=1}^{n} \oint_\gamma s_j.$$

Fix j, and write

$$s_j(z) = \sum_{k=1}^{\infty} a_{-k,j} (z - P_j)^{-k}.$$

Note that by definition, $a_{-1,j} = \mathrm{Res}_f(P_j)$. Since $\gamma([a, b])$ is compact and $P_j \notin \gamma([a, b])$, the series s_j converges uniformly in ζ on $\gamma([a, b])$. Therefore, commuting limits,

$$\oint_\gamma s_j = \sum_{k=1}^{\infty} a_{-k,j} \oint_\gamma (\zeta - P_j)^{-k} d\zeta.$$

We have seen that $(\zeta - P_j)^{-k}$ has a holomorphic antiderivative if $k \neq 1$ on $\mathbf{C} \setminus \{P_j\}$. Thus,

$$\oint_\gamma (\zeta - P_j)^{-k} d\zeta = 0$$

if $k \geq 2$, so

$$\oint_\gamma s_j = a_{-1,j} \oint_\gamma (\zeta - P_j)^{-1} d\zeta = 2\pi i \,\mathrm{Res}_f(P_j) \,\mathrm{Ind}_\gamma(P_j),$$

and the result is proven. □

Lemma 1.16.25. *Let f be a holomorphic function near P with a pole of order k at P. Then*

$$\mathrm{Res}_f(P) = \frac{1}{(k-1)!} \frac{\partial^{k-1}}{\partial z^{k-1}} \left[(z - P)^k f(z) \right] \Big|_{z=P}.$$

Proof. This is just the $j = -1$ case of **Proposition 1.16.17**. □

Theorem 1.16.26. *This theorem contains three parts:*

1. *Let f be holomorphic on an open set containing $H = \{z \mid \operatorname{Im} z \geq 0\}$ except for a finite number of singularities, $P_1, ..., P_n$, none of which are on the real axis. Suppose there are constants $M, p > 1$ and $R > 0$ such that $|f(z)| \leq \frac{M}{|z|^p}$ whenever $z \in H$ and $|z| \geq R$. Then*
$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{j=1}^{n} \operatorname{Res}_f(P_j).$$
2. *If the conditions of 1. hold with H replaced by $L = \{z \mid \operatorname{Im} z \leq 0\}$ (in particular, that the singularities $P_1, ..., P_n$ are in L), then*
$$\int_{-\infty}^{\infty} f(x)dx = -2\pi i \sum_{j=1}^{n} \operatorname{Res}_f(P_j).$$
3. *Conditions 1. and 2. hold if $f = \frac{P}{Q}$ where P and Q are polynomials, $\deg Q \geq \deg P + 2$, and Q has no real zeros.*

Proof. We tackle each claim one at a time.

1. Let $r > R$ and set $\gamma_r = \gamma_{1,r} + \gamma_{2,r}$, where $\gamma_{1,r}(t) = re^{it}$, $t \in [0, \pi]$, and $\gamma_{2,r}(t) = t$, $t \in [-r, r]$. Thus γ_r is a semicircle with base on the real axis, oriented counterclockwise.

   Choose r large enough so that all of the singularities of f lie inside γ_r. Then
   $$\oint_{\gamma_r} f dz = \int_{-r}^{r} f(t)dt + \int_0^{\pi} f\left(re^{i\theta}\right) ire^{i\theta} d\theta.$$
   We will show that
   $$\lim_{r\to\infty} \int_0^{\pi} f\left(re^{i\theta}\right) ire^{i\theta} d\theta = 0.$$
   Then it will follow from the fact that
   $$\oint_{\gamma_r} f dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}_f(P_j)$$
   that
   $$\int_{-\infty}^{\infty} = \lim_{r\to\infty} \int_{-r}^{r} f(t)dt = 2\pi i \sum_{j=1}^{n} \operatorname{Res}_f(P_j).$$
   Indeed, since $r > R$,
   $$\left|\int_0^{\pi} f\left(re^{i\theta}\right) ire^{i\theta} d\theta\right| \leq \int_0^{\pi} \frac{M}{r^p} r d\theta = \int_0^{\pi} \frac{M}{r^{p-1}} d\theta = \frac{M\pi}{r^{p-1}} \to 0$$
   as $r \to \infty$, since $p > 1$.

2. The argument here is exactly the same as in 1., except the orientation of $\gamma_{2,r}$ is now reversed, so that we maintain a positive, counterclockwise orientation on the semicircle.

3. We will show here that $|f(z)| \leq \frac{M}{|z|^2}$ for some M and $|z| > R$. Since P and Q are polynomials of degree d_1 and d_2, respectively, there exists $R > 0$ such that $|P(z)| \leq M_1|z|^{d_1}$. Similarly, $|Q(z)| \geq M_2|z|^{d_2}$ for $|z| > R$. Thus,
   $$|f(z)| = \frac{|P(z)|}{|Q(z)|} \leq \frac{M_1|z|^{d_1}}{M_2|z|^{d_2}} = \frac{M_1}{M_2}|z|^{d_1 - d_2}.$$
   Since $d_1 \leq d_2 - 2$, the proof is complete. □

Example 1.16.27. Compute $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^4}dx$.

We have, as in **Theorem 1.16.26** part 3, $P(x) = 1$ and $Q(x) = 1 + x^4$. Q has roots at $e^{i\frac{\pi}{4}}$, $e^{i\frac{3\pi}{4}}$, $e^{i\frac{5\pi}{4}}$, and $e^{i\frac{7\pi}{4}}$. Furthermore, $\deg Q - \deg P = 4 \geq 2$. The poles of f are simple, meaning of order 1.

So we need to compute the residues to evaluate the integral, as per **Theorem 1.16.26**. To do this, we turn to **Lemma 1.16.25**, which says that

$$\operatorname{Res}_f\left(e^{i\frac{\pi}{4}}\right) = \left.\frac{z - e^{i\frac{\pi}{4}}}{(1+z)^4}\right|_{z=e^{i\frac{\pi}{4}}} = \frac{1}{\left(e^{i\frac{\pi}{4}} - e^{i\frac{3\pi}{4}}\right)\left(e^{i\frac{\pi}{4}} - e^{i\frac{5\pi}{4}}\right)\left(e^{i\frac{\pi}{4}} - e^{i\frac{7\pi}{4}}\right)}.$$

And since $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$, we have that

$$\begin{aligned}
\operatorname{Res}_f\left(e^{i\frac{\pi}{4}}\right) &= \frac{1}{\left(e^{i\frac{\pi}{4}} - e^{i\frac{3\pi}{4}}\right)\left(e^{i\frac{\pi}{4}} - e^{i\frac{5\pi}{4}}\right)\left(e^{i\frac{\pi}{4}} - e^{i\frac{7\pi}{4}}\right)} \\
&= \frac{1}{e^{i\frac{\pi}{2}}\left(e^{-i\frac{\pi}{4}} - e^{i\frac{\pi}{4}}\right)e^{i\frac{3\pi}{4}}\left(e^{-i\frac{\pi}{2}} - e^{i\frac{\pi}{2}}\right)e^{i\pi}\left(e^{-i\frac{3\pi}{4}} - e^{i\frac{3\pi}{4}}\right)} \\
&= \frac{1}{-ie^{i\frac{3\pi}{4}}\left(2i\sin\frac{\pi}{4}\right)\left(2i\sin\frac{\pi}{2}\right)\left(2i\sin\frac{3\pi}{4}\right)} \\
&= \frac{-1}{8e^{-\frac{3\pi}{4}}\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}} \\
&= \frac{-1}{2\sqrt{2}(-1+i)}.
\end{aligned}$$

Note that we could compute the residue another way:

$$\operatorname{Res}_f\left(e^{i\frac{3\pi}{4}}\right) = \frac{1}{\left(e^{i\frac{3\pi}{4}} - e^{i\frac{\pi}{4}}\right)\left(e^{i\frac{3\pi}{4}} - e^{i\frac{5\pi}{4}}\right)\left(e^{i\frac{3\pi}{4}} - e^{i\frac{7\pi}{4}}\right)}.$$

Using this time the fact that $e^{i\theta} = \cos\theta + i\sin\theta$,

$$\begin{aligned}
\operatorname{Res}_f\left(e^{i\frac{3\pi}{4}}\right) &= \frac{1}{\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} - \cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} - \cos\frac{5\pi}{4} - i\sin\frac{5\pi}{4}\right)\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} - \cos\frac{7\pi}{4} - i\sin\frac{7\pi}{4}\right)} \\
&= \frac{-1}{2\sqrt{2}i(-1+i)}.
\end{aligned}$$

Note that these two residues are all we need to compute; our contour is the upper semicircle, and therefore we don't worry about the poles at $e^{i\frac{5\pi}{4}}$ and $e^{i\frac{7\pi}{4}}$. Said another way, since only $e^{i\frac{\pi}{4}}$ and $e^{i\frac{3\pi}{4}}$ are in H, these are the only residues we must compute.

See that $\operatorname{Res}_f\left(e^{i\frac{\pi}{4}}\right) + \operatorname{Res}_f\left(e^{i\frac{3\pi}{4}}\right) = \frac{-i}{2\sqrt{2}}$, so we can conclude that

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4}dx = 2\pi i \cdot \frac{-i}{2\sqrt{2}} = \frac{\pi}{\sqrt{2}}.$$

Example 1.16.28. For $b > 0$, we show that

$$I = \int_0^{\infty} \frac{\cos x}{x^2+b^2}dx = \frac{\pi e^{-b}}{2b}.$$

Observe first that the integrand is even, so

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos x}{x^2+b^2}dx.$$

We bring this problem into the realm of complex analysis. Let

$$g(z) = \frac{\cos z}{z^2 + b^2} = \frac{e^{iz} + e^{-iz}}{2(z^2 + b^2)}.$$

However, we have a problem with this construction. To compute the integral along the real axis, we need the integral over the arcs, whether they be the upper semicircle or the lower semicircle, to limit to zero. However, the numerator of g can get large whether we take the upper semicircle or the lower semicircle contour. See that $e^{iz} = e^{i(x+iy)} = e^{ix}e^{-y}$ which gets large if y is negative, i.e., z is in the lower half plane, and $e^{-iz} = e^{-i(x+iy)} = e^{-ix}e^{y}$ which gets large if y is positive, i.e., z is in the lower half plane.

Thus g will not work. To remedy this, take

$$f(z) = \frac{e^{iz}}{z^2 + b^2},$$

which as stated, only gets large in the lower half plane, so our contour will be the upper semicircle. Also,

$$\frac{\cos x}{x^2 + b^2} = \operatorname{Re}\left(\frac{e^{ix}}{x^2 + b^2}\right),$$

since $\cos x = \operatorname{Re} e^{ix}$ for $x \in \mathbf{R}$. So we will compute

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + b^2},$$

since the integral is on the real axis.

See that

$$f(z) = \frac{e^{iz}}{z^2 + b^2} = \frac{e^{iz}}{(z - ib)(z + ib)}$$

has simple poles at $z = \pm ib$. Since $z = -ib$ is outside of our curve, we disregard it, and $\operatorname{Res}_f(ib) = \frac{e^{-b}}{2ib}$.

Build our contour as before; $\gamma_R = \gamma_{1,R} + \gamma_{2,R}$, where $\gamma_{1,R}$ is the path along $\mathbf{R}$ from $-R$ to R and $\gamma_{2,R}$ is the counterclockwise upper semicircle connecting R to $-R$.

By choice of contour, the **Residue Theorem 1.16.24** gives that

$$\oint_{\gamma_R} f dz = 2\pi i \operatorname{Res}_f(ib) = 2\pi i \cdot \frac{e^{-b}}{2ib} = \frac{\pi e^{-b}}{b}.$$

We now turn to consider

$$\oint_{\gamma_{2,R}} \frac{e^{iz}}{z^2 + b^2} dz.$$

We know that $\gamma(t) = Re^{it}$, $t \in [0, \pi]$, and $\gamma'(t) = iRe^{it}$, so

$$\left|\oint_{\gamma_{2,R}} \frac{e^{iz}}{z^2 + b^2} dz\right| = \left|\int_0^{\pi} \frac{e^{iRe^{it}}}{R^2 e^{2it} + b^2} iRe^{it} dt\right| = \left|\int_0^{\pi} \frac{e^{iR(\cos t + i \sin t)}}{R^2\left(e^{2it} + \left(\frac{b}{R}\right)^2\right)} iRe^{it} dt\right| \leq \int_0^{\pi} \frac{e^{-R\sin t}}{R\left(1 - \left(\frac{b}{R}\right)^2\right)} dt \to 0$$

as $R \to \infty$, since $\left(1 - \frac{b^2}{R^2}\right) \geq \frac{1}{2}$ if R is large, and $e^{-R\sin t} < 1$. In fact, the convergence is uniform in $t \in [0, \pi]$ as $R \to \infty$. Therefore, we can conclude that

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + b^2} dx = \frac{1}{2}\oint_{\gamma_R} f dz = \frac{\pi e^{-b}}{2b}.$$

Example 1.16.29. We evaluate

$$I = \int_0^{2\pi} \frac{d\theta}{1+a^2-2a\cos\theta}$$

for $a > 0$ and $a \neq 1$. Note that this is the Poisson kernel for the Laplacian on the unit disk!

For this example, we want to view this integral on $\partial D(0,1)$. If so, $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$, so if $z = e^{i\theta}$, $\frac{1}{z} = e^{-i\theta}$, since we're on the unit circle. That suggests that we let

$$f(z) = \frac{1}{1+a^2-2a\left(\frac{1}{2}\left(z+\frac{1}{z}\right)\right)} \cdot \frac{1}{iz}.$$

One may wonder where the $\frac{1}{iz}$ term has come from (me too fella); it comes from the derivative of the parametrization of the curve. If we integrate $\oint_{\partial D(0,1)} h(z)dz = \int_0^{2\pi} g\left(e^{i\theta}\right) ie^{i\theta} d\theta$, the $\frac{1}{iz}$ cancels that $ie^{i\theta}$ term. It all works out in the end, I guess???

Now see that

$$\begin{aligned} f(z) &= \frac{1}{1+a^2-2a\left(\frac{1}{2}\left(z+\frac{1}{z}\right)\right)} \cdot \frac{1}{iz} \\ &= \frac{1}{1+a^2-az-\frac{a}{z}} \cdot \frac{1}{iz} \\ &= \frac{1}{i\left(-az^2+\left(1+a^2\right)z-a\right)} \\ &= \frac{1}{-i(z-a)(az-1)} \\ &= \frac{i}{(z-a)(az-1)} \\ &= \frac{i}{(z-a)\left(z-\frac{1}{a}\right)a}. \end{aligned}$$

There are simple poles at $z = a$ and $z = \frac{1}{a}$. Then $\mathrm{Res}_f(a) = \dfrac{i}{a^2-1}$ and $\mathrm{Res}_f\left(\dfrac{1}{a}\right) = \dfrac{i}{\left(\frac{1}{a}-a\right)a} = \dfrac{i}{1-a^2}$.

Therefore,

$$I = \begin{cases} \dfrac{2\pi}{1-a^2} & \text{if } 0 < a < 1; \\ \dfrac{2\pi}{a^2-1} & \text{if } a > 1. \end{cases}$$

Example 1.16.30. For $1 < a < 2$, we compute $\displaystyle\int_0^\infty \frac{x^{a-2}}{1+x^2}dx$.

Here, we let $f(z) = \dfrac{z^{a-2}}{1+z^2}$, as might be expected. Now, as $a \in \mathbf{R}$, by **Proposition 1.15.2**, we need to define a branch cut; thus, take f holomorphic on $\{z = x+iy \mid z \neq iy, y \leq 0\} = \left\{re^{i\theta} \mid r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2}\right\}$.

We build our contour as follows: let $\gamma_R = \gamma_{1,R} + \gamma_{2,R} + \gamma_{3,R} + \gamma_{4,R}$, where $\gamma_{1,R}$ goes from $-R$ to $\frac{-1}{R}$ on the real axis, $\gamma_{2,R}$ is the semicircle in the upper half plane from $\frac{-1}{R}$ to $\frac{1}{R}$, $\gamma_{3,R}$ goes from $\frac{1}{R}$ to R on the real axis, and $\gamma_{4,R}$ is the semicircle in the upper half plane from R to $-R$. Thus, γ_R is oriented positively and remains where f is holomorphic.

See now that

$$\lim_{R\to\infty} \oint_{\gamma_{3,R}} f dz = \int_0^\infty \frac{x^{a-2}}{1+x^2}dx,$$

the integral we seek.

To compute residues,

$$f(z) = \frac{z^{a-2}}{1+z^2} = \frac{z^{a-2}}{(z-i)(z+i)},$$

so $\mathrm{Res}_f(i) = \frac{1}{2i}\left(e^{i\frac{\pi}{2}}\right)^{a-2} = \frac{-1}{2i}\left(e^{ia\frac{\pi}{2}}\right)$. The residue at $-i$ is obviously not within our contour.

Thus by the **Residue Theorem 1.16.24**,

$$\oint_{\gamma_R} f dz = \frac{-1}{2i}\left(e^{ia\frac{\pi}{2}}\right) 2\pi i = -\pi e^{ia\frac{\pi}{2}}.$$

Now let's see what happens on each of the pieces $\gamma_{1,R}$, $\gamma_{2,R}$, and $\gamma_{4,R}$. What remains is the integral we desire (as $R \to \infty$).

Considering $\gamma_{1,R}$, we have

$$\oint_{\gamma_{1,R}} f dz = \int_{-R}^{\frac{-1}{R}} \frac{t^{a-2}}{1+t^2} dt = \int_{\frac{1}{R}}^{R} \frac{(-s)^{a-2}}{1+s^2} ds$$

via a substitution $s = -t$. Note that $(-s)^{a-2} = (-1)^{a-2}s^{a-2} = e^{i\pi(a-2)}s^{a-2} = e^{-\pi a}s^{a-2}$. So therefore,

$$\oint_{\gamma_{1,R}} f dz = \int_{\frac{1}{R}}^{R} \frac{(-s)^{a-2}}{1+s^2} = e^{i\pi a}\int_{\frac{1}{R}}^{R} \frac{s^{a-2}}{1+s^2} ds = e^{i\pi a}\oint_{\gamma_{3,R}} f dz.$$

Considering $\gamma_{2,R}$, we have that since $\gamma_{2,R}(t) = \frac{1}{R}e^{i(\pi-t)}$ for $t \in [0,\pi]$, $\gamma_{2,R}'(t) = \frac{-i}{R}e^{i(\pi-t)}$, so

$$\left|\oint_{\gamma_{2,R}} f dz\right| = \left|\int_0^{\pi} \frac{\left(\frac{1}{R}e^{i(\pi-t)}\right)^{a-2}}{1+\frac{1}{R^2}e^{2i(\pi-t)}}\left(\frac{-i}{R}\right)e^{i(\pi-t)} dt\right| \leq \int_0^{\pi} \frac{\frac{1}{R^{a-2}}}{\frac{1}{2}} \cdot \frac{1}{R} dt \to 0$$

as $R \to \infty$.

Considering $\gamma_{4,R}$, we have the same idea as we have seen; the integral will go to 0 as $R \to \infty$. Consider it a worthwhile exercise.

Therefore, we know that

$$\oint_{\gamma_R} f dz = -\pi e^{ia\frac{\pi}{2}},$$

and as $R \to \infty$, we have

$$\begin{aligned} -\pi e^{ia\frac{\pi}{2}} = \oint_{\gamma_R} f dz &= \left(1+e^{i\pi a}\right)\int_0^{\infty} \frac{t^{a-2}}{1+t^2} dt \\ &= e^{i\frac{\pi}{2}a}\left(e^{-i\frac{\pi}{2}a}+e^{i\frac{\pi}{2}a}\right)\int_0^{\infty} \frac{t^{a-2}}{1+t^2} dt \\ &= e^{i\frac{\pi}{2}a} 2\cos\left(\frac{\pi}{2}a\right)\int_0^{\infty} \frac{t^{a-2}}{1+t^2} dt, \end{aligned}$$

so

$$\frac{-\pi}{2\cos\left(\frac{\pi}{2}a\right)} = \int_0^{\infty} \frac{t^{a-2}}{1+t^2} dt.$$

Example 1.16.31. We compute $\displaystyle\int_0^{\infty} \frac{1}{x^2+6x+8} dx$.

We cannot just use $g(z) = \frac{1}{z^2+6z+8}$; given any contour which travels along the real axis, then along an arc of some angle θ, and then back to the origin, we see that $\frac{1}{(re^{i\theta})^2+6re^{i\theta}+8} \neq \frac{C}{r^2+6r+8}$, so no θ will work.

Instead, we let

$$f(z) = \frac{\log z}{z^2+6z+8},$$

with branch cut so that $\arg z \in (0, 2\pi)$ and consider the "Pac-man contour" given by $\gamma_R = \gamma_{1,R} + \gamma_{2,R} + \gamma_{3,R} + \gamma_{4,R}$, where

$$\begin{aligned}
\gamma_{1,R}(t) &= t + \frac{i}{\sqrt{2R}}, && t \in \left[\frac{1}{\sqrt{2R}}, R\right], \\
\gamma_{2,R}(t) &= Re^{it}, && t \in [\theta_0, 2\pi - \theta_0], \text{ where } \theta_0 = \theta_0(R) = \arcsin\left(\frac{1}{R\sqrt{2R}}\right), \\
\gamma_{3,R}(t) &= R - t - \frac{i}{\sqrt{2R}}, && t \in \left[0, R - \frac{1}{\sqrt{2R}}\right], \text{ and} \\
\gamma_{4,R}(t) &= \frac{e^{-it}}{\sqrt{R}}, && t \in \left[\frac{\pi}{4}, \frac{7\pi}{4}\right].
\end{aligned}$$

The ambitious reader can check that

$$\left|\oint_{\gamma_{2,R}} f dz\right| + \left|\oint_{\gamma_{4,R}} f dz\right| \to 0$$

as $R \to \infty$. See next that

$$\begin{aligned}
\oint_{\gamma_{1,R}} f dz = \int_{\frac{1}{\sqrt{2R}}}^{R} \frac{\log\left(t + \frac{i}{\sqrt{2R}}\right)}{\left(t + \frac{i}{\sqrt{2R}}\right)^2 + 6\left(t + \frac{i}{\sqrt{2R}}\right) + 8} \cdot 1 dt &= \int_{\frac{1}{\sqrt{2R}}}^{R} \frac{\log\left|t + \frac{i}{\sqrt{2R}}\right| + i \arg\left(t + \frac{i}{\sqrt{2R}}\right)}{\left(t + \frac{i}{\sqrt{2R}}\right)^2 + 6\left(t + \frac{i}{\sqrt{2R}}\right) + 8} dt \\
&\to \int_0^\infty \frac{\log t}{t^2 + 6t + 8} dt
\end{aligned}$$

as $R \to \infty$ (note that we ought to check that the convergence is uniform, another worthwhile exercise for the aspiring mathematician).

Also we have

$$\oint_{\gamma_{3,R}} f dz = \int_0^{R - \frac{1}{\sqrt{2R}}} \frac{\log\left(R - t - \frac{i}{\sqrt{2R}}\right)}{\left(R - t - \frac{i}{\sqrt{2R}}\right)^2 + 6\left(R - t - \frac{i}{\sqrt{2R}}\right) + 8} \cdot (-1) dt.$$

Now, via the substitution $s = R - t$, we get

$$\begin{aligned}
\int_0^{R - \frac{1}{\sqrt{2R}}} \frac{\log\left(R - t - \frac{i}{\sqrt{2R}}\right)}{\left(R - t - \frac{i}{\sqrt{2R}}\right)^2 + 6\left(R - t - \frac{i}{\sqrt{2R}}\right) + 8} \cdot (-1) dt &= \int_R^{\frac{1}{\sqrt{2R}}} \frac{\log\left(s - \frac{i}{\sqrt{2R}}\right)}{\left(s - \frac{i}{\sqrt{2R}}\right)^2 + 6\left(s - \frac{i}{\sqrt{2R}}\right) + 8} ds \\
&= -\int_{\frac{1}{\sqrt{2R}}}^{R} \frac{\log\left|s - \frac{i}{\sqrt{2R}}\right| + i \arg\left(s - \frac{i}{\sqrt{2R}}\right)}{\left(s - \frac{i}{\sqrt{2R}}\right)^2 + 6\left(s - \frac{i}{\sqrt{2R}}\right) + 8} ds \\
&\to -\int_0^\infty \left(\frac{\log s}{s^2 + 6s + 8} + \frac{2\pi i}{s^2 + 6s + 8}\right) ds
\end{aligned}$$

as $R \to \infty$.

For the residues,

$$f(z) = \frac{\log z}{z^2 + 6z + 8} = \frac{\log z}{(z+4)(z+2)},$$

so $\mathrm{Res}_f(-4) = \frac{\log(-4)}{-2} = \frac{\log 4 + i\pi}{-2}$ and $\mathrm{Res}_f(-2) = \frac{\log(-2)}{2} = \frac{\log 2 + i\pi}{2}$. Therefore,

$$-2\pi i \int_0^\infty \frac{1}{s^2 + 6s + 8} ds = 2\pi i \left(\frac{-1}{2} \log 4 + \frac{1}{2} \log 2\right),$$

so

$$\int_0^\infty \frac{1}{s^2 + 6s + 8} ds = \frac{1}{2} \log 2.$$

Example 1.16.32. We compute $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

To do so, let

$$f(z) = \frac{\pi \cot(\pi z)}{z^2}.$$

We still use the **Residue Theorem 1.16.24** here, but instead of isolating the integral we desire, we're going to isolate the sum of the residues.

For our contour, consider the box $\gamma_N = \gamma_{1,N} + \gamma_{2,N} + \gamma_{3,N} + \gamma_{4,N}$, where

$$\begin{aligned}
\gamma_{1,N}(t) &= N + \frac{1}{2} + it, && t \in \left[-\left(N+\frac{1}{2}\right), N + \frac{1}{2}\right],\\
\gamma_{2,N}(t) &= \left(N + \frac{1}{2} - t\right) + i\left(N + \frac{1}{2}\right), && t \in [0, 2N-1],\\
\gamma_{3,N}(t) &= -\left(N + \frac{1}{2}\right) + i\left(N + \frac{1}{2} - t\right), && t \in [0, 2N-1],\\
\gamma_{4,N}(t) &= -\left(N + \frac{1}{2}\right) i + t, && t \in \left[-\left(N+\frac{1}{2}\right), N + \frac{1}{2}\right].
\end{aligned}$$

See that $\cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)} = i\frac{e^{i\pi z}+e^{-i\pi z}}{e^{i\pi z}-e^{-i\pi z}}$.

We now ask: where do the residues of f occur? See clearly that $\sin(\pi z)$ is zero exactly when $y = 0$ and $x = k \in \mathbf{Z}$.

Note that it is a fact that away from $\mathbf{R} \subseteq \mathbf{C}$, $|\cot(\pi z)|$ is uniformly bounded in z. One may check that $|\cot(\pi z)| \leq \coth(\pi y)$ for $y > 0$. Thus, by the choice of the contour, $|\cot(\pi z)|$ is outright bounded, say by M, independently of N and of z. Thus,

$$\left| \oint_{\gamma_N} f dz \right| \leq \frac{4M(2N+1)}{N^2} \to 0$$

as $N \to \infty$. Therefore, by the **Residue Theorem 1.16.24**, the sum of the residues of f times $2\pi i$ must equal zero.

Now let's compute the residues. For $n \in \mathbf{Z} \setminus \{0\}$, we have

$$\begin{aligned}
\operatorname{Res}_f(n) = \lim_{z \to n} (z-n) \frac{\pi \cot(\pi z)}{z^2} &= \lim_{z\to n} \frac{\pi \cos(\pi z)}{z^2} \lim_{z \to n} \frac{z-n}{\sin(\pi z)}\\
&= \frac{\pi \cos(\pi n)}{n^2} \lim_{z\to n} \frac{z-n}{\sin(\pi z)}\\
&= \frac{\pi(-1)^n}{n^2} \lim_{z \to n} \frac{z-n}{\sin(\pi z) - \sin(\pi n)}\\
&= \frac{\pi(-1)^n}{n^2} \left(\frac{d}{dz} \sin(\pi z) \Big|_{z=n} \right)^{-1}\\
&= \frac{\pi(-1)^n}{n^2} \cdot \frac{1}{\pi \cos(\pi z)} \Big|_{z=n}\\
&= \frac{\pi(-1)^n}{n^2} \cdot \frac{1}{\pi(-1)^n}\\
&= \frac{1}{n^2}.
\end{aligned}$$

Thus is justified the choice of f.

At $z = 0$, however, f has a pole of order three. We compute:

$$\begin{aligned}
\operatorname{Res}_f(0) &= \frac{1}{2} \cdot \frac{\partial^2}{\partial z^2} \left[z^3 \frac{\pi \cos(\pi z)}{z^2 \sin(\pi z)} \right] \Big|_{z=0} \\
&= \frac{\pi}{2} \cdot \frac{\partial^2}{\partial z^2} \left[z \cot(\pi z) \right] \Big|_{z=0} \\
&= \frac{\pi}{2} \cdot \frac{\partial}{\partial z} \left[\cot(\pi z) - \pi z \csc^2(\pi z) \right] \Big|_{z=0} \\
&= \frac{\pi}{2} \left(-\pi \csc^2(\pi z) - \pi \csc^2(\pi z) + 2\pi^2 z \csc^2(\pi z) \cot(\pi z) \right) \Big|_{z=0} \\
&= \pi^2 \csc^2(\pi z)(\pi z \cot(\pi z) - 1) \Big|_{z=0} \\
&= \frac{\pi^2}{\sin^3(\pi z)} \left(\pi z \cos(\pi z) - \sin(\pi z) \right) \Big|_{z=0} \\
&= \frac{\pi^2}{\sin^3(\pi z)} \left(\left(\pi z - \frac{(\pi z)^3}{2} + \frac{(\pi z)^5}{4!} + \cdots \right) - \pi z + \frac{(\pi z)^3}{6} - \frac{(\pi z)^5}{5!} + \cdots \right) \Bigg|_{z=0} \\
&= \pi^2 \left(\frac{-1}{2} + \frac{1}{6} \right) \left(\frac{\pi z}{\sin(\pi z)} \right)^3 \Bigg|_{z=0} \\
&= \frac{-\pi^2}{3}.
\end{aligned}$$

But now,

$$-\operatorname{Res}_f(0) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \operatorname{Res}_f(n),$$

since all of the residues must sum to zero. This means that

$$\begin{aligned}
\frac{\pi^2}{3} &= 2 \sum_{n=1}^{\infty} \frac{1}{n^2}; \text{ i.e.,} \\
\frac{\pi^2}{6} &= \sum_{n=1}^{\infty} \frac{1}{n^2}.
\end{aligned}$$

Thus is solved the Basel problem.

Now, having practiced many examples using the **Residue Theorem 1.16.24** (a great use of time in studying for quals, perhaps?), we continue our discussion of not-quite-holomorphic functions.

Recall that a set $S \subseteq \mathbf{C}$ is **discrete 1.13.1** when every $z \in S$ has an $r > 0$ so that $S \cap D(z, r) = \{z\}$. We also say S consists of isolated points.

Definition 1.16.33. Let $U \subseteq \mathbf{C}$ be open. A **meromorphic** function on U with singular set S is a function $f : U \setminus S \to \mathbf{C}$ such that

1. S is discrete and closed,
2. f is holomorphic on $U \setminus S$, and
3. for each $z \in S$ and $r > 0$ with $D(z, r) \subseteq U$ and $D(z, r) \cap S = \{z\}$, the function $f|_{D(z,r)\setminus\{z\}}$ has a pole at z.

For convenience, we usually just say that f is a meromorphic function on U.

Lemma 1.16.34. *If $U \subseteq \mathbf{C}$ is a connected, open set, and $f \in H(U)$ but $f \not\equiv 0$, then the function $F : U \setminus Z_f \to \mathbf{C}$ defined by $F(z) = \frac{1}{f(z)}$, $z \in U \setminus Z_f$, is a meromorphic function with singular set Z_f. (Recall that $Z_f = \{z \mid f(z) = 0\}$.)*

Proof. Since f is continuous, $Z_f = f^{-1}(\{0\})$ is closed. Moreover, Z_f is discrete, by **Theorem 1.13.2**. F by construction is also holomorphic on $U \setminus Z_f$. Let $z_0 \in Z_f$, and choose $r > 0$ so that $D(z_0, r) \subseteq U$ and $D(z_0, r) \cap Z_f = \{z_0\}$. Then $F \in H(D(z_0, r) \setminus \{z_0\})$, $f(z_0) = 0$, and $|F(z)| = \frac{1}{|f(z)|} \to \infty$ as $z \to z_0$. Thus, F has a pole at z_0. Everything in the definition of **meromorphic 1.16.33** functions has been confirmed. $\square$

We can also extend our study of singularities to include the limiting behavior as $|z| \to \infty$.[5] Suppose $f : \mathbf{C} \to \mathbf{C}$ is entire. Then we can define $G(z) = f\left(\frac{1}{z}\right)$, and the behavior of G at 0 reflects the behavior of f as $|z| \to \infty$.

For example,

$$\lim_{|z| \to \infty} |f(z)| = \infty$$

if and only if G has a pole at 0.

We now make explicit this idea.

Definition 1.16.35. Suppose that $f : U \to \mathbf{C}$ is holomorphic on an open set $U \subseteq \mathbf{C}$ for which there exists $R > 0$ such that $\{z \mid |z| > R\} \subseteq U$. Define $G : \left\{z \mid 0 < |z| < \frac{1}{R}\right\} \to \mathbf{C}$ by $G(z) = f\left(\frac{1}{z}\right)$. Then

1. f has a **removable singularity at** ∞ if G has a removable singularity at 0,
2. f has a **pole at** ∞ if G has a pole at 0, and
3. f has an **essential singularity at** ∞ if G has an essential singularity at 0.

Note that we call U in the definition above a neighborhood of ∞.

The Laurent expansion of G around 0,

$$G(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

yields a series expansion for f that converges for $|z| > R$, namely,

$$f(z) = G\left(\frac{1}{z}\right) = \sum_{n=-\infty}^{\infty} a_n z^{-n} = \sum_{n=-\infty}^{\infty} a_{-n} z^n.$$

The series

$$\sum_{n=-\infty}^{\infty} a_{-n} z^n$$

is called the Laurent expansion of f around ∞.

Theorem 1.16.36. *Suppose $f \in H(\mathbf{C})$. Then $\lim_{|z| \to \infty} |f(z)| = \infty$ if and only if f is a nonconstant polynomial. The function f has a removable singularity at ∞ if and only if f is constant.*

Proof. Only one direction requires any work; certainly nonconstant polynomials limit to ∞, and if f is constant, it is bounded, and there is a removable singularity at ∞.

So for this nontrivial direction, since f is entire,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

for all $z \in \mathbf{C}$. Hence,

$$G(z) = f\left(\frac{1}{z}\right) = \sum_{n=-\infty}^{\infty} a_n z^{-n}$$

[5] Once on the **Riemann sphere 2.1.9** $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, this will be even more natural; ∞ is just another point there.

converges for all $z \in \mathbf{C} \setminus \{0\}$. The uniqueness of the Laurent series shows that this is the only possible Laurent series for G, so the Laurent expansion of f about ∞ is, not surprisingly,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

From the Laurent expansion for G, we learn that the function f has a pole at ∞ if and only if the Laurent expasion for f has a finite number of terms of positive powers. The function f has a removable singularity at ∞ if the expansion has only the constant term, i.e., $f(z) = a_0$. □

1.17 The Zeros of a Holomorphic Function

Definitions: zero of order k, argument principle, simple point
Main Idea: We use the argument principle to count the orders of the poles and zeros of a meromorphic function inside a disk. We then prove the Open Mapping Theorem,and discover that if a holomorphic function takes a value with order k, then it takes all nearby values with order k. Rouché's Theorem lets us count zeros in a disk if we can compare the function to a function with known zeros, Hurwitz discusses normal limits of nonvanishing holomorphic functions, and the Maximum Modulus results tell us where a holomorphic function achieves maxima and minima.

In this section, we try to characterize some answers to the following questions:

Does a nonconstant holomorphic function on an open set have an open image? (Yes, the **Open Mapping Theorem 1.17.7**.)

What conditions on the local geometry of a holomorphic function might force it to be constant? (Among other things, we have the **Maximum Modulus Principle 1.17.15**.)

We start with the idea of counting zeros and poles. To introduce this concept, suppose $U \subseteq \mathbf{C}$ is an open, connected set, and $f \in H(U)$. Say also that $\overline{D(z_0, r)} \subseteq U$. We know that the value of f in $D(z_0, r)$ is determined by its values on $\partial D(z_0, r)$, by the **Cauchy Integral Formula 1.9.3**. We would like to know how many zeros f has in $D(z_0, r)$.

So say f has a zero at z'. Then

$$f(z) = \sum_{j=k}^{\infty} \frac{1}{j!} \frac{\partial^j f}{\partial z^j}(z')(z - z')^j,$$

for some $k \geq 1$.

Definition 1.17.1. In this situation, we say that f has a **zero of order** k (or of multiplicity k) at z'. If $n = 1$, we say that f has a simple zero at z'.

So the definition feels just like poles of order k and simple poles.

Lemma 1.17.2. *If f is holomorphic on a neighborhood of $\overline{D(z_0, r)}$ and has a zero of order n at z_0 and no other zeros in $\overline{D(z_0, r)}$, then*

$$\frac{1}{2\pi i} \oint_{\partial D(z_0, r)} \frac{f'(\zeta)}{f(\zeta)} d\zeta = n.$$

Proof. By hypothesis, f having a zero of order n at z_0 means that

$$f(z) = \sum_{j=n}^{\infty} \frac{1}{j!} \frac{\partial^j f}{\partial z^j}(z_0)(z - z_0)^j = (z - z_0)^n H(z),$$

where

$$H(z) = \sum_{j=n}^{\infty} \frac{1}{j!} \frac{\partial^j f}{\partial z^j}(z_0)(z - z_0)^{j-n}$$

is a holomorphic function on a neighborhood of $\overline{D(z_0,r)}$. Also, by construction,

$$H(z_0) = \frac{1}{n!}\frac{\partial^n f}{\partial z^n}(z_0) \neq 0.$$

Next, for $\zeta \in \overline{D(z_0,r)} \setminus \{z_0\}$, we have by the product rule that

$$\frac{f'(\zeta)}{f(\zeta)} = \frac{(\zeta - z_0)H'(\zeta) + n(\zeta - z_0)^{n-1}H(\zeta)}{(\zeta - z_0)^n H(\zeta)} = \frac{H'(\zeta)}{H(\zeta)} + n(\zeta - z_0)^{-1}.$$

By continuity, f is nowhere zero away from z_0, so H is as well. It follows that $\frac{H'}{H}$ is holomorphic on a neighborhood of $\overline{D(z_0,r)}$. Therefore,

$$\oint_{|\zeta - z_0|=r} \frac{f'(\zeta)}{f(\zeta)} d\zeta = \oint_{|\zeta - z_0|=r} \frac{H'(\zeta)}{H(\zeta)} d\zeta + \oint_{|\zeta - z_0|=r} \frac{n}{\zeta - z_0} d\zeta = 2\pi i n,$$

by the **Cauchy Intgral Theorem 1.9.4** and by **Lemma 1.9.2**, respectively. $\square$

Lemma 1.17.3. *Suppose $U \subseteq \mathbf{C}$ is open, and $f \in H(U)$. Say $\overline{D(P,r)} \subseteq U$, and f is nonvanishing on $\partial D(P,r)$. Let $z_1, ..., z_k$ be the zeros of f in $D(P,r)$. Let n_ℓ be the order of the zero of f at z_ℓ. Then*

$$\frac{1}{2\pi i}\oint_{\partial D(P,r)} \frac{f'(\zeta)}{f(\zeta)} d\zeta = \sum_{\ell=1}^{k} n_\ell.$$

Note that this is really just a step further from **Lemma 1.17.2**. The next few results will continue this trend, building up to a way to count all zeros and poles of a function.

Proof. Set

$$H(z) = \frac{f(z)}{(z - z_1)^{n_1} \cdots (z - z_k)^{n_k}}$$

for $z \in U \setminus \{z_1, ..., z_k\}$. Then for each $j \in \{1, ..., k\}$,

$$H(\zeta) = \frac{f(\zeta)}{(\zeta - z_j)^{n_j}} \cdot \prod_{\ell \neq j} \frac{1}{(\zeta - z_\ell)^{n_\ell}}.$$

The second term is holomorphic near z_j, and the first has a removable singularity at z_j, as in the proof of **Lemma 1.17.2**. Thus, H is holomorphic where f is, namely, on a neighborhood of $\overline{D(P,r)}$.

Calculating as in the proof of **Lemma 1.17.2**,

$$\frac{f'(\zeta)}{f(\zeta)} = \frac{H'(\zeta)}{H(\zeta)} + \sum_{\ell=1}^{k} \frac{n_\ell}{\zeta - z_\ell}.$$

The function $\frac{H'}{H}$ is holomorphic on an open set containing $\overline{D(P,r)}$, so

$$\oint_{|\zeta - P|=r} \frac{f'(\zeta)}{f(\zeta)} d\zeta = 0 + \sum_{\ell=1}^{k} n_\ell \oint_{|\zeta - P|=r} \frac{1}{\zeta - z_\ell} d\zeta = 2\pi i \sum_{\ell=1}^{k} n_\ell.$$

$\square$

Definition 1.17.4. We call the formula

$$\frac{1}{2\pi i}\oint_{|\zeta - P|=r} \frac{f'(\zeta)}{f(\zeta)} d\zeta = \sum_{\ell=1}^{k} n_\ell$$

the **argument principle**. It, of course, is a way to count the total order of zeros of f inside $D(P,r)$.

Let's investigate the argument principle a bit closer. Set $\gamma(t) = f\left(P + re^{it}\right)$, $t \in [0, 2\pi]$. Then, since f is holomorphic, $\gamma'(t) = f'\left(P + re^{it}\right) ire^{it}$, so

$$\frac{1}{2\pi i} \int_0^{2\pi} \frac{\gamma'(t)}{\gamma(t)} dt = \frac{1}{2\pi i} \frac{f'\left(P + re^{it}\right)}{f\left(P + re^{it}\right)} ire^{it} dt = \frac{1}{2\pi i} \oint_{|\zeta - P| = r} \frac{f'(\zeta)}{f(\zeta)} d\zeta.$$

The expression

$$\frac{1}{2\pi i} \int_0^{2\pi} \frac{\gamma'(t)}{\gamma(t)} dt$$

is the **index 1.16.22** of the curve γ around 0, which intuitively, gives the number of times the f-image of $\partial D(P, r)$ winds around $0 \in \mathbf{C}$.

The argument principle, counting the orders of zeros inside a disk, extends to meromorphic functions as well, counting in addition the orders of poles.

Lemma 1.17.5. *If $f : U \setminus \{Q\} \to \mathbf{C}$ is a nowhere zero holomorphic function on $U \setminus \{Q\}$ with a pole of order n at Q, and if $\overline{D(Q, r)} \subseteq U$, then*

$$\frac{1}{2\pi i} \oint_{\partial D(Q,r)} \frac{f'(\zeta)}{f(\zeta)} d\zeta = -n.$$

Proof. From the comment right after the definition of **pole of order** k **1.16.15**, we know $(z-Q)^n f(z)$ has a removable singularity at Q. Set $H(z) = (z - Q)^n f(z)$ on $\overline{D(Q, r)} \setminus \{Q\}$, so that H extends to a holomorphic function on a neighborhood of $\overline{D(Q, r)}$.

For $z \in \overline{D(Q, r)} \setminus \{Q\}$, the product rule says that

$$\frac{H'(z)}{H(z)} = \frac{n(z - Q)^{n-1} f(z) + (z - Q)^n f'(z)}{(z - Q)^n f(z)}.$$

Since $\frac{H'}{H}$ is holomorphic on $\overline{D(Q, r)}$,

$$\frac{1}{2\pi i} \oint_{|\zeta - Q| = r} \frac{H'(\zeta)}{H(\zeta)} d\zeta = 0$$

by the **Cauchy Integral Theorem 1.9.4**. Therefore, as before,

$$\oint_{|\zeta - Q| = r} \frac{f'(\zeta)}{f(\zeta)} d\zeta = -n \oint_{|\zeta - Q| = r} \frac{1}{\zeta - Q} d\zeta = -2\pi i n.$$

□

Lemmas 1.17.2, **1.17.3**, and **1.17.5** may all feel very familiar in their statements and proofs. This is by design! We have explored the argument principle cautiously, so that the following theorem, which consolidates all our results, does not appear to come from thin air. Pedagogically, we took baby steps up to this theorem, so that we could better appreciate it.

Theorem 1.17.6 (The Argument Principle for Meromorphic Functions). *Suppose f is a meromorphic function on an open set $U \subseteq \mathbf{C}$ so that $\overline{D(Q, r)} \subseteq U$, and f has neither zeros nor poles on $\partial D(Q, r)$. Then*

$$\frac{1}{2\pi i} \oint_{\partial D(Q,r)} \frac{f'(\zeta)}{f(\zeta)} d\zeta = \sum_{j=1}^{p} n_j - \sum_{k=1}^{q} m_k,$$

where $n_1, ..., n_p$ are the multiplicities of the zeros $z_1, ..., z_p$ of f in $D(Q, r)$ and $m_1, ..., m_q$ are the orders of the poles $w_1, ..., w_q$ of f in $D(Q, r)$.

Proof. The proof follows the outline of the previous Lemmas, as stated. On a neighborhood of $\overline{D(Q,r)} \setminus \{w_1, ..., w_q\}$, set

$$H(z) = \frac{(z-w_1)^{m_1} \cdots (z-w_q)^{m_q}}{(z-z_1)^{n_1} \cdots (z-z_p)^{n_p}} f(z).$$

By the earlier arguments, H extends to a nonvanishing holomorphic function on a neighborhood of $\overline{D(Q,r)}$. Moreover, observe that

$$\frac{H'(z)}{H(z)} = -\sum_{j=1}^{p} \frac{n_\ell}{z-z_\ell} + \sum_{k=1}^{q} \frac{m_k}{z-w_k} + \frac{f'(z)}{f(z)}.$$

Since $\frac{H'}{H}$ is holomorphic near $\overline{D(Q,r)}$, we have

$$\oint_{|\zeta-Q|=r} \frac{f'(\zeta)}{f(\zeta)} d\zeta = 0 + \sum_{j=1}^{p} n_\ell \oint_{|\zeta-Q|=r} \frac{1}{\zeta - z_\ell} d\zeta - \sum_{k=1}^{q} m_k \oint_{|\zeta-Q|=r} \frac{1}{\zeta-w_k} d\zeta = 2\pi i \left(\sum_{j=1}^{p} n_j - \sum_{k=1}^{q} m_k \right),$$

as desired. □

We now answer one of the questions posed at the beginning of this section. Nonconstant holomorphic functions are open maps! Note that the same is *not* true in **R** (as is the case every time we compare to **R**); $f(x) = \sin x$ is analytic, but $f((-\pi,\pi)) = [-1,1]$, which is not open.

Theorem 1.17.7 (**The Open Mapping Theorem**). *If $U \subseteq \mathbf{C}$ is a connected, open set, and $f : U \to \mathbf{C}$ is a nonconstant holomorphic function, then $f(U)$ is an open set.*

Proof. We need to show that given $Q \in f(U)$, there exists a disk $D(Q,\varepsilon) \subseteq f(U)$.

Let $P \in U$ with $f(P) = Q$. Set $g(z) = f(z) - Q$. Then $g(P) = 0$, and g is nonconstant. Consequently, there exists $r > 0$ so that $\overline{D(P,r)} \subseteq U$ and g does not vanish on $\overline{D(P,r)} \setminus \{P\}$. Suppose g vanishes to order n at P for some $n \geq 1$. Then by the **Argument Principle for Meromorphic Functions 1.17.6**, we have

$$\frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{f'(\zeta)}{f(\zeta)-Q} d\zeta = \frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{g'(\zeta)}{g(\zeta)} d\zeta = n.$$

The nonvanishing of $g(\zeta)$ on $\partial D(P,r)$ and the compactness of $\partial D(P,r)$ means that there exists $\varepsilon > 0$ such that $|g(\zeta)| > \varepsilon$ on $\partial D(P,r)$. We claim that for this ε, $D(Q,\varepsilon) \subseteq f(U)$. Since Q is an arbitrary point of $f(U)$, this claim suffices to prove the theorem.

To prove the claim, set

$$N(z) = \frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{f'(\zeta)}{f(\zeta)-z} d\zeta$$

for $z \in D(Q,\varepsilon)$. Note that $f(\zeta) - z$ does not vanish on $\partial D(P,r)$, since if $z \in D(Q,\varepsilon)$,

$$|f(\zeta) - z| \geq |f(\zeta) - Q| - |z - Q| > \varepsilon - |z-Q| > 0.$$

Thus, N is a continuous (in fact, C^∞) function of $z \in D(Q,\varepsilon)$. Also, N counts the zeros of $f(\bullet) - z$ in $D(P,r)$ by the **Argument Principle1.17.6**. Smooth, integer valued functions must be constant, so since $N(Q) = n$, this forces $N(z) = n$ for all $z \in D(Q,\varepsilon)$.

Since $n \geq 1$, we see that for each fixed $z \in D(Q,\varepsilon)$, the function $g(\zeta) = f(\zeta) - z$ vanishes at some point or points of $D(P,r)$- i.e., f takes each value of $z \in D(Q,\varepsilon)$ with multiplicity n. Therefore, the claim is proven. □

Note that in the proof of the **Open Mapping Theorem 1.17.7**, we observed that if f takes the value Q at the point P with multiplicity k, then f behaves locally much like the function $\varphi(z) = Q + (z-P)^k$. We'll explore this.

Definition 1.17.8. A **simple point of** f is a point q so that $f(z)-q$ vanishes to order 1. (A multiple point of f is a value $q \in \operatorname{Range} f$ so that $f(z)-q$ vanishes to at least second order.)

Lemma 1.17.9. *Let $f : U \to \mathbf{C}$ be a nonconstant holomorphic function on a connected, open set $U \subseteq \mathbf{C}$. Then the multiple points of f are isolated.*

Proof. Since f is nonconstant, the holomorphic function f' is not identically 0. This means that the zeros of f' are isolated. Multiple points q of f have the property that $f'(p) = 0$ when $f(p) = q$. Indeed,

$$f(z) - p = \sum_{n=2}^{\infty} a_n (z-p)^n$$

if f vanishes to order at least 2 at p. Thus, multiple points are isolated, as desired. $\square$

Theorem 1.17.10. *Suppose that $f : U \to \mathbf{C}$ is a nonconstant holomorphic function on a connected, open set $U \subseteq \mathbf{C}$, such that $P \in U$ and $f(P) = Q$ with order k. Then, there are numbers $\delta, \varepsilon > 0$ so that each $q \in D(Q, \varepsilon) \setminus \{Q\}$ has exactly k distinct preimages in $D(P, \delta)$, and each preimage is a simple point of f.*

Proof. By **Lemma 1.17.9**, there exists $\delta_1 > 0$ such that every point $f(D(P, \delta_1) \setminus \{P\})$ is a simple point of f. Now, choose $\delta, \varepsilon > 0$ so that $0 < \delta < \delta_1$, $Q \notin f(D(P, \delta) \setminus \{P\})$, $D(Q, \varepsilon) \subseteq f(D(P, \delta))$, and $D(Q, \varepsilon) \cap f(\partial D(P, \delta)) = \emptyset$. Such an ε exists, because the **Open Mapping Theorem 1.17.7** asserts that f is an open map.

Thus, we have seen that

$$\frac{1}{2\pi i} \oint_{\partial D(P,\delta)} \frac{f'(\zeta)}{f(\zeta) - q} d\zeta = k > 0,$$

as the function $\frac{f'(\zeta)}{f(\zeta)-q}$ is smooth in q and locally constant.

By the **Argument Principle 1.17.6**, if $P_1, ..., P_\ell$ are the zeros of $f(\bullet) - q$ in $D(P, \delta)$ with orders $n_1, ..., n_\ell$, then $n_1 + \cdots + n_\ell = k$.

By the choice of $\delta < \delta_1$, each $n_j = 1$ for $j \in \{1, ..., \ell\}$, so it follows that $\ell = k$.

Thus $f^{-1}(\{q\})$ has exactly k preimages $P_1, ..., P_k$, each of which is a simple point of f. $\square$

Let's make a few remarks about **Theorem 1.17.10**:

1. This theorem does not assert that the image of a disk is a disk, but instead, that the image of a disk about P contains a disk about $f(P)$.
2. If $f'(P) \neq 0$, then f is locally one-to-one, and hence locally invertible, with $f^{-1}(w)$ the unique preimage of w in $D(P, \delta)$. Now,

Lemma 1.17.11. *f^{-1} is holomorphic.*

Proof. We first show that f^{-1} is continuous at Q. f is an open map, and $f = \left(f^{-1}\right)^{-1}$. Thus, f^{-1} is continuous.

We next show that f^{-1} is complex differentiable at q near Q. Say $f(p) = q$. Then, we need to show that

$$\lim_{q' \to q} \frac{f^{-1}(q') - f^{-1}(q)}{q' - q}$$

exists. Observe that

$$\frac{f^{-1}(q') - f^{-1}(q)}{q' - q} = \frac{1}{\frac{q' - q}{f^{-1}(q') - f^{-1}(q)}} = \frac{1}{\frac{f(p') - f(p)}{p' - p}}.$$

We know that

$$\lim_{p' \to p} \frac{f(p') - f(p)}{p' - p} = f'(p) \neq 0.$$

The continuity of f^{-1} implies that $p' \to p$ as $q' \to q$; thus

$$\lim_{q' \to q} \frac{f^{-1}(q') - f^{-1}(q)}{q' - q}$$

exists, and is $\frac{1}{f'(p)}$. $\square$

Note that **Lemma 1.17.11** was indeed the **Inverse Function Theorem 1.15.4** for holomorphic mappings!

We now continue our discussion of the zeros of holomorphic functions, now with a focus on what the global behavior of the functions must be.

Theorem 1.17.12 (**Rouché's Theorem**). *Suppose that f and g are holomorphic on U and $U \subseteq \mathbf{C}$ is open. Suppose further that $\overline{D(z_0, r)} \subseteq U$, and for each $\zeta \in \partial D(z_0, r)$,*

$$|f(\zeta) + g(\zeta)| < |f(\zeta)| + |g(\zeta)|.$$

Then

$$\frac{1}{2\pi i} \oint_{\partial D(z_0,r)} \frac{f'(\zeta)}{f(\zeta)} d\zeta = \frac{1}{2\pi i} \oint_{\partial D(z_0,r)} \frac{g'(\zeta)}{g(\zeta)} d\zeta.$$

As these are argument principles of holomorphic functions, this means that the number of zeros of f and g in $D(z_0, r)$ is the same (counting multiplicity). A few more comments:

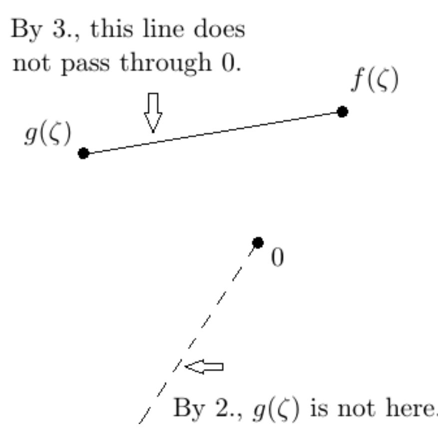
1. One of the requirements of the application of the argument principle is that your function be nonvanishing on the boundary of the disk. The hypotheses of **Rouché 1.17.12** ensure this; if either f or g vanish on $\partial D(z_0, r)$, then we have equality, not strict inequality, in $|f(\zeta) + g(\zeta)| < |f(\zeta)| + |g(\zeta)|$. Therefore, $\frac{f'}{f}$ and $\frac{g'}{g}$ are indeed integrable on $\partial D(z_0, r)$.

2. Furthermore, the inequality also implies that $\frac{f(\zeta)}{g(\zeta)}$ is not real and negative on $\partial D(z_0, r)$. To see this, suppose $\frac{f(\zeta)}{g(\zeta)} = \lambda \leq 0$ for some $\zeta \in \partial D(z_0, r)$. Then

$$\left| \frac{f(\zeta)}{g(\zeta)} - 1 \right| = |\lambda - 1| = -\lambda + 1 = \left| \frac{f(\zeta)}{g(\zeta)} \right| + 1,$$

which implies that $|f(\zeta) - g(\zeta)| = |f(\zeta)| + |g(\zeta)|$, and since g and $-g$ have the same number of zeros, this contradicts the hypothesized inequality.

3. Finally, the same argument also shows that $tf(\zeta) + (1-t)g(\zeta) \neq 0$ for any $t \in [0, 1]$. (The graph below should help clarify.)



Let us now prove **Rouché's Theorem 1.17.12**.

Proof. Let $f_t(z) = tf(z)+(1-t)g(z)$. For a fixed $t \in [0,1]$, f_t is holomorphic on U. Observe that $f_0(z) = g(z)$, $f_1(z) = f(z)$, and by 3. above, $f_t(\zeta) \neq 0$ for any $t \in [0,1]$ and $\zeta \in \partial D(z_0, r)$. Thus, we may let

$$I_t = \frac{1}{2\pi i} \oint_{\partial D(z_0,r)} \frac{{f_t}'(\zeta)}{f_t(\zeta)} d\zeta.$$

We know by the **Argument Principle 1.17.6** that I_t is an integer, and for any fixed t, $f_t(z)$ is continuous in t, ${f_t}'(z)$ is continuous in t, and $f_t(\zeta)$ is bounded away from 0 for all $t \in [0,1]$ and $\zeta \in \partial D(z_0, r)$. Thus, the integrand is bounded and continuous in t, and thus I_t is continuous in t. Therefore, I_t is constant, so $I_0 = I_1$, which proves the claim. ☐

Example 1.17.13. Let $f(z) = z^7 + 5z^3 - z - 2$; we determine the number of roots of $f(z)$ in $D(0,1)$. The trick to using **Rouché 1.17.12** is to find a function with known roots that satisfies the hypotheses; in other words, we must come up with the g.

Here, let $g(z) = 5z^3$. Then, if $\zeta \in \partial D(0,1)$, we have

$$|f(\zeta) - g(\zeta)| = \zeta^7 - \zeta - 2| \leq |\zeta|^7 + |\zeta| + 2 = 4 < 5 = |g(\zeta)| \leq |f(\zeta)| + |g(\zeta)|.$$

So since g has, with multiplicity, three zeros in $D(0,1)$, by **Rouché's Theorem 1.17.12**, so does f.

Note as a fun aside that we can use **Rouché's Theorem 1.17.12** to prove the **Fundamental Theorem of Algebra 1.11.6**:

Proof. Let $P(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_2 z^2 + a_1 z + a_0$. Let $g(z) = z^n$. Now, if $r > 1$, and $\zeta \in \partial D(0,r)$, then

$$|P(\zeta) - g(\zeta)| = |a_{n-1}\zeta^{n-1} + \cdots + a_0| \leq |a_{n-1}|r^{n-1} + \cdots + |a_0| \leq (|a_{n-1}| + \cdots + |a_0|)\, r^{n-1}.$$

So, if $r > 1 + |a_{n-1}| + \cdots + |a_0|$, then by **Rouché's Theorem 1.17.12**, $P(z)$ has n zeros in $D(0,r)$. Thus, $P(z)$ has n zeros with multiplicity in $\mathbf{C}$. ☐

Theorem 1.17.14 (Hurwitz's Theorem). *Let $U \subseteq \mathbf{C}$ be connected and open, and let $f_j : U \to \mathbf{C}$ be holomorphic and nonvanishing. If (f_j) converges uniformly on compact subsets of U to f_0, then either $f_0(z) \equiv 0$ for all $z \in U$, or $f_0(z) \neq 0$ for all $z \in U$.*

Proof. Suppose $f_0(z_0) = 0$ for some $z_0 \in U$, but $f_0 \not\equiv 0$. Then, there exists $r > 0$ such that $f_0(z) \neq 0$ for all $z \in D(z_0, r) \setminus \{z_0\}$. Then

$$\frac{1}{2\pi i} \oint_{\partial D\left(z_0, \frac{r}{2}\right)} \frac{{f_0}'(\zeta)}{f_0(\zeta)} d\zeta > 0$$

is a positive integer that counts the multiplicity of the zero of f_0 at z_0.

But, if $1 \leq j < \infty$,

$$\frac{1}{2\pi i} \oint_{\partial D\left(z_0, \frac{r}{2}\right)} \frac{{f_j}'(\zeta)}{f_j(\zeta)} d\zeta = 0.$$

But we know from **Corollary 1.12.3** that ${f_j}'(\zeta) \to {f_0}'(\zeta)$ uniformly on $\partial D\left(z_0, \frac{r}{2}\right)$, and $\frac{1}{f_j(\zeta)} \to \frac{1}{f_0(\zeta)}$ uniformly on $\partial D\left(z_0, \frac{r}{2}\right)$ since f_0 is bounded away from 0 on $\partial D\left(z_0, \frac{r}{2}\right)$. Therefore, we commute the limits to see that

$$0 = \frac{1}{2\pi i} \oint_{\partial D\left(z_0, \frac{r}{2}\right)} \frac{{f_j}'(\zeta)}{f_j(\zeta)} \to \frac{1}{2\pi i} \oint_{\partial D\left(z_0, \frac{r}{2}\right)} \frac{{f_0}'(\zeta)}{f_0(\zeta)} d\zeta > 0,$$

a contradiction. ☐

Note, *again*, the real variable situation is different. For example, $f_j(x) = x^2 + \frac{1}{j}$ is zero free for all j on $\mathbf{R}$, but the uniform limit $f_j \to x^2$ is not. (Note that this is not a counter example to **Hurwitz 1.17.14**, because $f_j(z) = z^2 + \frac{1}{j}$ is *not* zero free on $D(0,1)$.

We now begin to answer the second question posed at the start of this section: "What conditions on the local geometry of a holomorphic function might force it to be constant?"

Theorem 1.17.15 (The Maximum Modulus Principle). *Let $U \subseteq \mathbf{C}$ be a connected, open set, and let $f \in H(U)$. If there exists $z_0 \in U$ such that $|f(z_0)| \geq |f(z)|$ for all $z \in U$, then f is constant.*

Proof. Assume not. Then f is not constant, so $f(U)$ is an open set, by the **Open Mapping Theorem 1.17.7**. This means that for some $r > 0$, $D(f(z_0), r) \subseteq f(U)$. Consequently, there exists $w \in D(f(z_0), r) \subseteq f(U)$ so that $|w| > |f(z_0|$. This is a contradiction. $\square$

Corollary 1.17.16 (The Maximum Modulus Theorem). *Let $U \subseteq \mathbf{C}$ be a bounded, connected, and open set. Let $f \in C(\overline{U}) \cap H(U)$. Then*

$$\max_{\overline{U}} |f| = \max_{\partial U} |f|.$$

Proof. Since $|f|$ is a continuous function, its max must occur somewhere, say at z_0. If $z_0 \in \partial U$, then we are done. If $z_0 \in U$, then by the **Maximum Modulus Principle 1.17.15**, f is constant on $\overline{U}$, and hence the max also occurs on ∂U. $\square$

Truly, we have proven something a bit stronger:

Theorem 1.17.17. *Let $U \subseteq \mathbf{C}$ be a connected, open set, and let $f \in H(U)$. If there is a point $z_0 \in U$ at which $|f|$ has a local maximum, then f is constant.*

Also, note that boundedness is an important property to have for the **Maximum Modulus Theorem 1.17.16**; consider the following (non) example:

Example 1.17.18. Let $U = \left\{ z = x + iy \mid \frac{-\pi}{2} < y < \frac{\pi}{2} \right\}$, connected, open, and unbounded. Let $f(z) = e^{e^z}$. Then on ∂U, where $y = \pm\frac{\pi}{2}$, we see that $e^z = e^{x \pm i\frac{\pi}{2}} = \pm i e^x$, so on ∂U, $\left| e^{e^z} \right| = 1$. However, $\lim\limits_{x \to \infty} e^{e^x} = \infty$.

Towards the end of semester two in **Theorem 3.2.5**, we'll see exactly how fast a function has to grow to beat the **Maximum Modulus Theorem 1.17.16** on an unbounded set; essentially, this e^{e^z} function characterizes the growth rate.

We also have a "minimum modulus principle," of sorts:

Lemma 1.17.19. *Let f be holomorphic on a connected, open set $U \subseteq \mathbf{C}$. Assume that f never vanishes. If there is a point $z_0 \in U$ at which $|f(z_0)| \leq |f(z)|$ for all $z \in U$, then f is constant.*

Proof. Apply the **Maximum Modulus Principle 1.17.15** to $g(z) = \frac{1}{f(z)}$. $\square$

Note that **Lemma 1.17.19** implies that on bounded domains, 0 is the only allowable local minimum of $|f(z)|$.

1.18 The Schwarz Lemma

Definitions:
Main Idea: Schwarz and Schwarz-Pick are precursors to discussing conformal maps of the unit disk. Indeed, rotations (Schwarz) and Möbius transformation (Schwarz-Pick) are the only such maps. Look forward to semester two!

The next two results tend to be very useful.

Lemma 1.18.1 (**Schwarz**)**.** *Let $f \in H(D(0,1))$. Assume that $|f(z)| \le 1$ for all $z \in D(0,1)$ and that $f(0) = 0$. Then $|f(z)| \le |z|$, and $|f'(0)| \le 1$. If either $|f(z)| = |z|$ for some $z \neq 0$, or $|f'(0)| = 1$, then f is a rotation; i.e., $f(z) = \alpha z$ for some $\alpha \in \mathbf{C}$, $|\alpha| = 1$.*

Proof. Consider the function $g(z) = \frac{f(z)}{z}$. We see that $g \in H(D(0,1) \setminus \{0\})$, and

$$\lim_{z\to 0} g(z) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = f'(0).$$

If, therefore, we extend $g(z)$ to be $f'(0)$ when $z = 0$, then $g \in C(D(0,1))$, and by **Riemann Removable Singularities 1.16.2**, $g \in H(D(0,1))$.

For each $\varepsilon > 0$ small, consider $\overline{D(0, 1-\varepsilon)}$. On $\partial D(0,1-\varepsilon)$, $|f(z)| \le 1$ implies that $|g(z)| \le \frac{1}{1-\varepsilon}$. By the **Maximum Modulus Theorem 1.17.16**, this yields that $|g(z)| \le \frac{1}{1-\varepsilon}$ for all $z \in D(0, 1-\varepsilon)$. Sending $\varepsilon \to 0^+$ yields that $|g(z)| \le 1$ for all $z \in D(0,1)$. This means $|f(z)| \le |z|$. Also, $|g(0)| = |f'(0)| \le 1$.

Finally, assume that $|f(z)| = |z|$ for some $z \neq 0$. Then $|g(z)| = 1$. Since $|g(w)| \le 1$ for all $w \in D(0,1)$, it follows from the **Maximum Modulus Principle 1.17.15** that $g(z)$ is a constant of modulus 1, say α. Thus, $f(z) = \alpha z$. If $|f'(0)| = 1$, then $|g(0)| = 1$, and by the same argument, $g(z)$ is a constant of modulus 1, say α. This means $f(z) = \alpha z$. ☐

Relaxing the requirement in **Schwarz's Lemma 1.18.1** that $f(0) = 0$ gives us the following.

Theorem 1.18.2 (**Schwarz-Pick**)**.** *Let $f : D(0,1) \to \overline{D(0,1)}$ be holomorphic. Then for any $a \in D(0,1)$ and with $b = f(a)$, we have the estimate*

$$|f'(a)| \le \frac{1 - |b|^2}{1 - |a|^2}.$$

Moreover, if $f(a_1) = b_1$ and $f(a_2) = b_2$, then

$$\left| \frac{b_2 - b_1}{1 - \overline{b_1} b_2} \right| \le \left| \frac{a_2 - a_1}{1 - \overline{a_1} a_2} \right|.$$

Proof. If $c \in D(0,1)$, then set

$$\varphi_c(z) = \frac{z - c}{1 - \bar{c} z}.$$

Observe that

$$\begin{aligned} \left| \frac{z-c}{1 - \bar{c}z} \right| < 1 &\text{ if and only if } |z - c|^2 < |1 - \bar{c}z|^2, \\ &\text{if and only if } |z|^2 - 2\operatorname{Re}(\bar{z}c) + |c|^2 < 1 - 2\operatorname{Re}(c\bar{z}) + |c|^2|z|^2, \\ &\text{if and only if } |c|^2\left(1 - |z|^2\right) < 1 - |z|^2, \\ &\text{if and only if } |z| < 1, \end{aligned}$$

since $|c| < 1$. Thus, for every $c \in D(0,1)$, $\varphi_c : D(0,1) \to D(0,1)$.[6]

Also,

$$\begin{aligned} w = \frac{z - c}{1 - \bar{c}z} &\text{ if and only if } w - \bar{c}zw = z - c, \\ &\text{if and only if } z(1 + \bar{c}w) = w + c, \\ &\text{if and only if } z = \frac{w + c}{1 + \bar{c}w}. \end{aligned}$$

Thus, ${\varphi_c}^{-1}$ exists, and ${\varphi_c}^{-1} = \varphi_{-c}$.

[6] Pay special attention to this family of functions φ_c; they are Möbius transformations, which we will see a lot more of as the notes progress!

Next, for f as in the hypotheses, set $g(z) = \varphi_b(f(\varphi_{-a}(z)))$. Then $g : D(0,1) \to \overline{D(0,1)}$ is holomorphic and $g(0) = \varphi_b(f(\varphi_{-a}(0))) = \varphi_b(f(a)) = \varphi_b(b) = 0$. Therefore, by **Schwarz's Lemma 1.18.1**, $|g'(0)| \leq 1$.

We now compute, by iterated chain rule,

$$g'(z) = \varphi_b{}'(f(\varphi_{-a}(z)))f'(\varphi_{-a}(z))\varphi_{-a}{}'(z).$$

See that a general φ_c has derivative, by the quotient rule,

$$\varphi_c{}'(z) = \frac{(1-\bar{c}z) + \bar{c}(z-c)}{(1-\bar{c}z)^2} = \frac{1-|c|^2}{(1-\bar{c}z)^2}.$$

Now notice that $\varphi_{-a}{}'(0) = 1 - |a|^2$, and $\varphi_b{}'(b) = \frac{1}{1-|b|^2}$, so

$$1 \geq \frac{1}{1-|b|^2}|f'(a)|\left(1-|a|^2\right).$$

Rearranging this inequality gives us the first conclusion.

For the second conclusion, set $h(z) = \varphi_{b_1}(f(\varphi_{-a_1}(z)))$. Then **Schwarz's Lemma 1.18.1** implies that $|h(z)| \leq |z|$; i.e., $|\varphi_{b_1}(f(\varphi_{-a_1}(z)))| \leq |z|$.

With $w = \varphi_{-a_1}(z)$, we have $|\varphi_{b_1}(f(w))| \leq |\varphi_{a_1}(w)|$. Taking $w = a_2$ yields $|\varphi_{b_1}(b_2)| \leq |\varphi_{a_1}(a_2)|$, which, when written out explicitly, is equivalent to the second condition. $\square$

2 Complex II

Though the notes for semester one continue for two more days, this is a natural point to introduce the second semester's topics. The second semester deals with a lot more topological concepts; in particular, much of it is spent on **conformal mappings 2.1.1** and the **Riemann Mapping Theorem 2.2.2**, and then normal convergence, harmonic and subharmonic functions, factorization theorems, and more.

2.1 Holomorphic Functions as Geometric Maps

Definitions: conformal map, Möbius transformation, fractional linear transformation (on $\mathbf{C}$), Riemann sphere, fractional linear transformation (on $\widehat{\mathbf{C}}$), limit of a sequence (in $\widehat{\mathbf{C}}$), inverse Cayley transform
Main Idea: In this section, we define conformal maps, describe all conformal self maps on $\mathbf{C}$ (linear), describe all conformal self maps on $D(0,1)$ (Möbius transformation and rotation), and describe all conformal self maps on $\widehat{\mathbf{C}}$ (fractional linear transformations).

Definition 2.1.1. Let $U, V \subseteq \mathbf{C}$ be open. Let $h : U \to V$ be holomorphic, one-to-one, and onto. Then we say that h is **conformal**, or that h is **biholomorphic**.

Note a few facts:

1. If h is one-to-one, then h' is never 0. Thus h has no **multiple points 1.17.8**.
2. A conformal map can transfer holomorphic functions on U to V, and vice versa. For example, $f \in H(V)$ if and only if $f \circ h \in H(U)$. Similarly, $g \in H(U)$ if and only if $g \circ h^{-1} \in H(V)$.

Now, a most natural question to ask is: given a connected, open set, what types of conformal maps exist on it? We'll first answer this question for conformal self maps on $\mathbf{C}$.

Lemma 2.1.2. *If $f : \mathbf{C} \to \mathbf{C}$ is conformal, then $\lim_{|z|\to\infty} |f(z)| = \infty$, in the sense that given $\varepsilon > 0$, there exists $C > 0$ such that if $|z| > C$, then $|f(z)| > \frac{1}{\varepsilon}$.*

Proof. The set $\overline{D\left(0, \frac{1}{\varepsilon}\right)}$ is a compact subset of $\mathbf{C}$. Since f^{-1} is holomorphic, it is continuous, and therefore $S = f^{-1}\left(\overline{D\left(0, \frac{1}{\varepsilon}\right)}\right)$ is a compact subset of $\mathbf{C}$. By Heine-Borel, S is therefore bounded. Thus, there exists $C > 0$ satisfying $S \subseteq D(0, C)$. This means that if $|w| > C$, then $w \notin f^{-1}\left(D\left(0, \frac{1}{\varepsilon}\right)\right)$, which in turn, means that $|f(w)| > \frac{1}{\varepsilon}$. ☐

Theorem 2.1.3. *A function $f : \mathbf{C} \to \mathbf{C}$ is a conformal mapping if and only if there exist $a, b \in \mathbf{C}$ with $a \neq 0$ so that $f(z) = az + b$.*

Proof. For the noninteresting direction, suppose $f(z) = az + b$, $a \neq 0$. Then $f \in H(\mathbf{C})$, and f is invertible, as if $w = az + b$, then $z = \frac{w-b}{a}$. The function f is clearly a bijection.

For the other direction, fix an arbitrary conformal map $f : \mathbf{C} \to \mathbf{C}$. By **Lemma 2.1.2**, f has a pole at ∞; there exists $C > 0$ so that $|f(z)| > 1$ if $|z| > C$. This means that the function

$$g(z) = \frac{1}{f\left(\frac{1}{z}\right)}$$

is defined on $D\left(0, \frac{1}{C}\right) \setminus \{0\}$, and $|g(z)| \leq 1$. By **Riemann Removable Singularities 1.16.2**, g extends to a holomorphic function on $D\left(0, \frac{1}{C}\right)$, and $g(0) = 0$, since f has a pole at ∞.

Next, since $f : \mathbf{C} \to \mathbf{C}$ is one-to-one, g is one-to-one on $D\left(0, \frac{1}{C}\right)$. Also, $g(z) \neq 0$ unless $z = 0$. Since g is one-to-one, $g'(0) \neq 0$. And since

$$0 \neq |g'(0)| = \lim_{z\to 0} \left|\frac{g(z) - g(0)}{z - 0}\right| = \lim_{z\to 0} \left|\frac{g(z)}{z}\right|,$$

there exists $A > 0$ so that $|g(z)| > A|z|$ if z is sufficiently small. We now claim that there exist numbers $B, D > 0$ so that if $|z| > D$, then $|f(z)| < B|z|$. Indeed, since there exists $\delta > 0$ so that $|g(z)| > A|z|$ if $|z| < \delta$, this means that if $|z| > \frac{1}{\delta}$, then

$$|f(z)| = \frac{1}{\left|g\left(\frac{1}{z}\right)\right|} < \frac{1}{A\left|\frac{1}{z}\right|} = \frac{1}{A}|z|,$$

so the claim holds with $B = \frac{1}{A}$ and $D = \frac{1}{\delta}$. By **Theorem 1.11.5**, f is a polynomial of degree at most 1. This means $f(z) = az + b$ for some $a, b \in \mathbf{C}$, and since f is one-to-one, $a \neq 0$. The result is shown. $\square$

Lemma 2.1.4. *A holomorphic function $f : D(0,1) \to D(0,1)$ that satisfies $f(0) = 0$ is a conformal self map of $D(0,1)$ if and only if there exists $\omega \in \mathbf{C}$, $|\omega| = 1$, such that $f(z) = \omega z$ for all $z \in D(0,1)$.*

Note that since $|\omega| = 1$, $\omega = e^{i\theta}$ for $\theta \in [0, 2\pi)$, a rotation.

Proof. For the boring direction, if $\omega \in \mathbf{C}$, $|\omega| = 1$, then $f(z) = \omega z$ is a conformal self map with inverse $w \mapsto \frac{w}{\omega}$.

We now assume that $f : D(0,1) \to D(0,1)$ is conformal, and that $f(0) = 0$. Let $g = f^{-1}$. By **Schwarz's Lemma 1.18.1**, $|f'(0)| \leq 1$ and $|g'(0)| \leq 1$. Since $z = f(g(z))$, by the chain rule $1 = f'(g(z))g'(z)$. So $1 = f'(0)g'(0)$. It thus follows that $|f'(0)| = |g'(0)| = 1$. The **Schwarz Lemma 1.18.1** now forces $f(z) = f'(0)z$, and the lemma is verified with $\omega = f'(0)$. $\square$

To find all conformal self maps of $D(0,1)$, we take a clue from the proof of the **Schwarz-Pick Lemma 1.18.2** (indeed, we mentioned to look out for the following):

Definition 2.1.5. For $a \in \mathbf{C}$, $|a| < 1$, define $\varphi_a(z) = \frac{z-a}{1-\overline{a}z}$. Then φ_a is called a **Möbius transformation**.

Lemma 2.1.6. *If φ_a is a Möbius transformation, then φ_a is a conformal self map of $D(0,1)$.*

Proof. First, we must show that φ_a is holomorphic for all $z \in D(0,1)$, i.e., that $1 - \overline{a}z \neq 0$. But notice that $1 - \overline{a}z = 0$ implies that $z = \frac{1}{\overline{a}} \notin D(0,1)$. Thus $\varphi_a \in H(D(0,1))$.

Next,

$$\begin{aligned}
\left|\frac{z-a}{1-\overline{a}z}\right| < 1 &\text{ if and only if } |z-a| < |1-\overline{a}z| \\
&\text{ if and only if } |z-a|^2 < |1-\overline{a}z|^2 \\
&\text{ if and only if } |z|^2 - z\overline{a} - \overline{z}a + |a|^2 < 1 - \overline{a}z - a\overline{z} + |\overline{a}z|^2 \\
&\text{ if and only if } |z|^2 + |a|^2 < 1 + |a|^2|z|^2 \\
&\text{ if and only if } |z|^2\left(1 - |a|^2\right) < 1 - |a|^2 \\
&\text{ if and only if } |z|^2 < 1.
\end{aligned}$$

Thus, φ_a maps $D(0,1)$ to $D(0,1)$.

Also,

$$\begin{aligned}
w = \frac{z-a}{1-\overline{a}z} &\text{ if and only if } w - \overline{a}zw = z - a \\
&\text{ if and only if } z(1 + \overline{a}w) = w + a \\
&\text{ if and only if } z = \varphi_{-a}(w).
\end{aligned}$$

Since $|a| < 1$, $|-a| < 1$, so φ_a is invertible on $D(0,1)$ with inverse φ_{-a}. The lemma is proven. $\square$

Theorem 2.1.7. *Let $f \in H(D(0,1), D(0,1))$. Then f is a conformal self map of $D(0,1)$ if and only if there exist $a, \omega \in \mathbf{C}$ with $|a| < 1$ and $|\omega| = 1$ such that $f(z) = \omega\varphi_a(z)$ for all $z \in D(0,1)$.*

Proof. Since the composition of two conformal self maps is a conformal self map, we know from **Lemmas 2.1.4** and **2.1.6** that $f(z) = \omega\varphi_a(z)$ is a conformal self map of $D(0,1)$ if $|\omega| = 1$ and $|a| < 1$.

For the interesting direction, assume $f : D(0,1) \to D(0,1)$ is an arbitrary conformal self map. Set $b = f(0)$, and observe that the conformal self map $\varphi_b \circ f : D(0,1) \to D(0,1)$ satisfies $\varphi_b(f(0)) = \varphi_b(b) = 0$. By **Lemma 2.1.4**, there exists $\omega \in \mathbf{C}$ with $|\omega| = 1$ so that $\varphi_b \circ f(z) = \omega z$ for $z \in D(0,1)$. This means

$$f(z) = {\varphi_b}^{-1} \circ \varphi_b \circ f(z) = \varphi_{-b}(\omega z) = \frac{\omega z + b}{1 + \bar{b}\omega z} = \omega \frac{z + \frac{b}{\omega}}{1 + \bar{b}\omega z}.$$

Since $\omega = e^{i\theta}$ for some $\theta \in [0, 2\pi)$, $\omega^{-1} = e^{-i\theta} = \overline{\omega}$. Consequently, if $a = -b\omega^{-1}$, then $f(z) = \omega\varphi_a(z)$, as desired. $\square$

We now explore Möbius transformations in more generality, and see how they yield other conformal maps.

Definition 2.1.8. A function of the form $f(z) = \frac{az+b}{cz+d}$ is called a **fractional linear transformation** if $ad - bc \neq 0$.

Note some of the following remarks:

1. If $z \mapsto \frac{az+b}{cz+d}$, and $ad - bc = 0$, then the numerator and denominator are multiples, and the resulting map is a constant.

2. Even when $ad - bc \neq 0$, f is still not defined for all z. In particular, f is undefined when $z = \frac{-d}{c}$, and
$$\lim_{z \to \frac{-d}{c}} \left| \frac{az+b}{cz+d} \right| = \infty;$$
i.e., there is a simple pole at $\frac{-d}{c}$.

3. Furthermore,
$$\lim_{|z| \to \infty} \frac{az+b}{cz+d} \begin{cases} = \frac{a}{c} & \text{if } c \neq 0; \\ \text{does not exist} & \text{if } c = 0. \end{cases}$$
Though, if $c = 0$, $f(z) = \frac{a}{d}z + \frac{b}{d}$, and we have said all there is about such maps in **Theorem 2.1.3**.

To remedy the fact that fractional linear transformations are not always defined, mathematicians add a point at ∞ and consider the value of $\frac{az+b}{cz+d}$ to be ∞ when $z = \frac{-d}{c}$. From this point of view, $f : \mathbf{C} \cup \{\infty\} \to \mathbf{C} \cup \{\infty\}$. One may think of this point topologically; this is the one point compactification of a plane, or formally, with specific regard to the behavior of ∞, as we do in this class. Regardless, the intuition is clear: we have transformed problems on the plane into problems on the sphere.

Definition 2.1.9. The construction $\mathbf{C} \cup \{\infty\}$ is called the **Riemann sphere**, often written concisely as $\widehat{\mathbf{C}}$, with the topology described momentarily.

Thus, on the Riemann sphere, we redefine fractional linear transformations.

Definition 2.1.10. A function $f : \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ is a **fractional linear transformation** if there exist $a, b, c, d \in \mathbf{C}$, $ad - bc \neq 0$, such that

1. if $c = 0$, $f(\infty) = \infty$ and $f(z) = \frac{a}{d}z + \frac{b}{d}$ for all $z \in \mathbf{C}$, or

2. if $c \neq 0$, $f(\infty) = \frac{a}{c}$, $f\left(\frac{-d}{c}\right) = \infty$, and $f(z) = \frac{az+b}{cz+d}$ for $z \in \mathbf{C} \setminus \left\{\frac{-d}{c}\right\}$.

Rather than continuously treat the point at infinity as a formal addition and specify where every functions maps to ∞ and what ∞ is mapped to, we now describe the topology necessary to treat the Riemann sphere as a sincere compactification of $\mathbf{C}$.

Definition 2.1.11. A sequence $(p_j) \subseteq \widehat{\mathbf{C}}$ **converges** to $p_0 \in \widehat{\mathbf{C}}$, written

$$\lim_{j \to \infty} p_j = p_0,$$

if either

1. $p_0 = \infty$ and $\lim_{j\to\infty} |p_j| = \infty$, where the limit is taken for all j such that $p_j \in \mathbf{C}$ (or there are only finitely many $p_j \in \mathbf{C}$), or

2. $p_0 \in \mathbf{C}$, all but finitely many of the p_j are in $\mathbf{C}$, and $\lim_{j\to\infty} p_j = p_0$, taken in the usual sense.

Now, note that stereographic projection establishes a one-to-one correspondence between $S^2 = \{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 + z^2 = 1\} \subseteq \mathbf{R}^3$ and $\widehat{\mathbf{C}}$ in such a way that convergence of the sequence is preserved in both directions. As such, it is hopefully now clear why $\widehat{\mathbf{C}}$ is often referred to as a sphere.

Theorem 2.1.12. *If $f : \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ is a fractional linear transformation, then f is a one-to-one, onto, and continuous function. Moreover, $f^{-1} : \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ is also a fractional linear transformation. Lastly, if $g : \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ is another fractional linear transformation, then so is $f \circ g$.*

This theorem demonstrates that the set of all fractional linear transformations is a group under the operation of function composition!

Proof. To first see that f is invertible, see that if $w = \frac{az+b}{cz+d}$, then $czw + dw = az + b$, so $z = \frac{dw-b}{-cw+a}$.

Also, f is holomorphic away from $z = \frac{-d}{c}$, so it is continuous there.

We define

$$g(w) = \begin{cases} \frac{dw-b}{-cw+a} & \text{if } w \neq \frac{a}{c}, \infty; \\ \frac{-d}{c} & \text{if } w = \infty; \\ \infty & \text{if } w = \frac{a}{c}. \end{cases}$$

Then g is a fractional linear transfromation, and it is now easy to check that $f \circ g$ and $g \circ f$ are both the indentity on $\widehat{\mathbf{C}}$- by all means, do so.

Finally, if $g(z) = \frac{Az+B}{Cz+D}$, then the composition is

$$g \circ f(z) = \frac{A\left(\frac{az+b}{cz+d}\right) + B}{C\left(\frac{az+b}{cz+d}\right) + D} = \frac{Aaz + Ab + Bcz + Bd}{Caz + Cb + Dcz + Dd} = \frac{(Aa + Bc)z + (Ab + Bd)}{(Ca + Dc)z + (Cb + Dd)},$$

and

$$\begin{aligned} &(Aa + Bc)(Cb + Dd) - (Ab + Bd)(Ca + Dc) \\ &\quad = ACab + ADad + BCbc + BDcd - ACab - ADbc - BCad - BDcd \\ &\quad = (AD - BC)(ad - bc) \\ &\quad \neq 0, \end{aligned}$$

since f and g are fractional linear transformations.

The composition properties and continuity also hold at ∞. □

Proposition 2.1.13. *The function φ is a conformal self map of $\widehat{\mathbf{C}}$ if and only if φ is a fractional linear transformation.*

Note that there are a few special fractional linear transformations. Here is one commonly used:

Definition 2.1.14. The **inverse Cayley transform** is $\varphi(z) = \frac{z-i}{z+i}$.[7]

Theorem 2.1.15. *The inverse Cayley transform maps the upper half plane, $\{z \mid \operatorname{Im} z > 0\}$, conformally onto the unit disk.*

[7] Not according to Wikipedia. Also the book itself contradicts this; page 21 versus page 189.

Proof. φ is a conformal self map of the Riemann sphere by **Proposition 2.1.13**, so it only remains to check the restriction maps to the correct codomain, a straightforward calculation. See that

$$\begin{aligned}
\left|\frac{z-i}{z+i}\right| < 1 \text{ implies } |z-i|^2 < |z+i|^2,\\
\text{which implies } (z-i)(\overline{z}+i) < (z+i)(\overline{z}-i),\\
\text{which implies } |z|^2 + iz - i\overline{z} + 1 < |z^2 - iz + i\overline{z} + 1,\\
\text{which implies } 4\operatorname{Re}(iz) < 0,\\
\text{which implies } \operatorname{Im} z > 0.
\end{aligned}$$

□

2.2 The Riemann Mapping Theorem

Definitions: homeomorphic, normal convergence, bounded on compact sets, holomorphically simply connected

Main Idea: The Riemann Mapping Theorem says that if U is simply connected, then it is conformally equivalent to $D(0,1)$. In this section, we also talk about normal convergence, prove Montel's Theorem, and talk about holomorphic simple connectivity and its consequences, like holomorphic logarithms and square roots.

The **Riemann Mapping Theorem 2.2.2** is one of the landmark achievements in the theory of one complex variable. It characterizes the open sets that are conformally equivalent to the unit disk.

Note that there are a few restrictions:

1. U cannot be all of $\mathbf{C}$. This is because if $f : \mathbf{C} \to D(0,1)$, then f is bounded, and by **Liouville 1.11.4**, must be constant.
2. Furthermore, conformal maps are homeomorphisms, since a holomorphic function is continuous, so if U is conformally equivalent to $D(0,1)$, U is homeomorphic to $D(0,1)$.

Recall that

Definition 2.2.1. Two open sets U and V are **homeomorphic** if there is a one-to-one, onto, and continuous function $f : U \to V$ with f^{-1} continuous also. We say that f is a homeomorphism.

Remarkably, these are the only conditions!

Theorem 2.2.2 (The Riemann Mapping Theorem). *If $U \subseteq \mathbf{C}$ is an open set, $U \neq \mathbf{C}$, and U is homeomorphic to the unit disk, then U is conformally equivalent to the unit disk.*

The proof of the **Riemann Mapping Theorem 2.2.2** has two parts.

In **Part I**, we show that if U is homeomorphic to $D(0,1)$, then each holomorphic function on U has a holomorphic antiderivative.

In **Part II**, we first construct a one-to-one holomorphic map from U into $D(0,1)$. Then we use the existence of that map to find a holomorphic bijection, by solving an extremal problem.

Both parts need to be delayed so that we can build up some important machinery. We will do **Part II** first, and then **Part I**.

Note also that the **Riemann Mapping Theorem 2.2.2** only holds for sets homeomorphic to $D(0,1)$. As we will see in **Theorem 2.4.22**, the annuli $\{z \mid r_1 < |z| < r_2\}$ and $\{z \mid s_1 < |z| < s_2\}$ are conformally equivalent if and only if $\frac{r_2}{r_1} = \frac{s_2}{s_1}$. So the **Riemann Mapping Theorem 2.2.2** easily fails when U is not homeomorphic to $D(0,1)$.

The idea of solving an extremal problem to prove the **Riemann Mapping Theorem 2.2.2** actually comes from the proof of **Schwarz's Lemma 1.18.1**. There, we saw that if $f : D(0,1) \to D(0,1)$ was holomorphic and $f(0) = 0$, then f is conformal if and only if $|f'(0)|$ is as large as possible; i.e., $|f'(0)| = \sup\{|h'(0)| \mid h : D(0,1) \to D(0,1), h(0) = 0, h \in H(D(0,1))\}$. The idea is to look at $f \in H(U, D(0,1))$, and maximize $|f'(p)|$ for some $p \in U$. There are some techinical issues which we will need to address when we get there, like

1. Is $\sup\{|h'(p)| \mid h \in H(U, D(0,1)), h(p) = 0\}$ finite?
2. Is the supremum achieved? (This was actually not checked by Riemann, so we can gloat about being smarter than him.)

So we now turn toward tools that will help us understand the proof of the **Riemann Mapping Theorem 2.2.2**, the first of which is normal families. Normal families will provide the machinery that establishes the existence of the extremal function in the proof of the **Riemann Mapping Theorem 2.2.2**.

Definition 2.2.3. A sequence of functions (f_j) defined on an open set $U \subseteq \mathbf{C}$ **converges normally** to a limit function f_0 on U if (f_j) converges to f_0 uniformly on compact subsets of U.

In other words, convergence is normal if for each compact $K \subseteq U$ and each $\varepsilon > 0$, there exists $N = N(K, \varepsilon)$ such that if $n \geq N$ and $z \in K$, then $|f_n(z) - f_0(z)| < \varepsilon$.

Example 2.2.4. Here's an example in $\mathbf{R}$: let $U = (0,1)$ and $f_n(x) = x^n$. Then $(f_n) \to 0$ normally.

Definition 2.2.5. Let $\mathcal{F}$ be a family of function on an open set $U \subseteq \mathbf{C}$. We say that $\mathcal{F}$ is **bounded on compact sets** if for each compact $K \subseteq U$, there is a constant $M = M(K) > 0$ such that for all $f \in \mathcal{F}$ and $z \in K$, $|f(z)| \leq M$.

Theorem 2.2.6 (**Montel's Theorem**)**.** *Let $U \subseteq \mathbf{C}$, and let $\mathcal{F}$ be a family of holomorphic functions that is bounded on compact sets. Then, for every sequence $(f_j) \subseteq \mathcal{F}$, there is a subsequence that converges normally on U to a limit function f, necessarily holomorphic.*

(This is basically Arzela-Ascoli in $\mathbf{C}$.)

Proof. The proof is outlined as follows:

0. Select a countable set of points $C = \{z_1, ...\}$ which are dense in U; i.e., $U \subseteq \overline{C}$.
1. Show that there is a subsequence of (f_j) that converges at all points of C.
2. Use the density of C to establish that the subsequence converges on all of U, and check that the convergence is normal.

So, let us proceed:

0. Let $C = \{z_j \mid j \in \mathbf{N}\}$ be an enumeration of the points in U with rational real and imaginary parts.
1. Let $(f_j) \subseteq \mathcal{F}$ be any sequence. We use a diagonalization argument. The singleton set $\{z_1\}$ is compact, so there exists M_1 so that $|f_j(z_1)| \leq M_1$ for all j. Thus, there exists a subsequence of $(f_j(z_1))$ that converges to a point $w_1 \in \mathbf{C}$, $|w_1| \leq M_1$. Relabel this subsequence $f_{1,1}(z_1)$, $f_{1,2}(z_1)$, $f_{1,3}(z_1)$, ...

   Next, the set $\{z_2\}$ is compact, so there exists M_2 so that $|f_{1,j}(z_2)| \leq M_2$. Thus, there exists a subsequence of $(f_{1,j}(z_2))$ that converges, say to w_2. Relabel this subsequence $f_{2,1}(z_2)$, $f_{2,2}(z_2)$, ...

   Continuing in this process yields an array:

$$\begin{array}{ccccccccc}
f_{1,1}(z_1), & f_{1,2}(z_1), & f_{1,3}(z_1), & f_{1,4}(z_1), & f_{1,5}(z_1), & \dots & \to & w_1 \\
f_{2,1}(z_2), & f_{2,2}(z_2), & f_{2,3}(z_2), & f_{2,4}(z_2), & f_{2,5}(z_2), & \dots & \to & w_2 \\
f_{3,1}(z_3), & f_{3,2}(z_3), & f_{3,3}(z_3), & f_{3,4}(z_3), & f_{3,5}(z_3), & \dots & \to & w_3 \\
f_{4,1}(z_4), & f_{4,2}(z_4), & f_{4,3}(z_4), & f_{4,4}(z_4), & f_{4,5}(z_4), & \dots & \to & w_4 \\
f_{5,1}(z_5), & f_{5,2}(z_5), & f_{5,3}(z_5), & f_{5,4}(z_5), & f_{5,5}(z_5), & \dots & \to & w_5 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & & \vdots
\end{array}$$

In this array, the kth horizontal row converges to $w_k \in \mathbf{C}$, and we have that $\{f_{k,j} \mid 1 \leq j \leq \infty\} \subseteq \{f_{k-1,j} \mid 1 \leq j \leq \infty\}$.

Set $g_n = f_{n,n}$. Then $(g_n)_{n=1}^{\infty} \subseteq (f_j)_{j=1}^{\infty}$, and $(g_n)_{n=k}^{\infty} \subseteq (f_{k,\ell})_{\ell=1}^{\infty}$. This means that

$$\lim_{n\to\infty} g_n(z_k) = w_k$$

for all $k \in \mathbf{N}$.

2. Now, we use the density of C to show that g_n converges on U.

   Let $K \subseteq U$. We claim that, given $\varepsilon > 0$, there exists $\delta > 0$ so that if $\zeta, \xi \in K$ and $|\zeta - \xi| < \delta$, then $|g_\ell(\zeta) - g_\ell(\xi)| < \varepsilon$ for all $\ell \in \mathbf{N}$. (Note that this property is called equicontinuity.)

   To prove this claim, it suffices to prove it on closed disks. Suppose $E = \overline{D(z_0, r)} \subseteq U$. Since E is closed, there exists $\rho > 0$ so that $E \subseteq D(z_0, \rho) \subseteq U$. By hypothesis, there exists $M_{D\left(z_0, \frac{r+\rho}{2}\right)}$ so that $|g_n(z)| \leq M_{D\left(z_0, \frac{r+\rho}{2}\right)}$ for all $n \in \mathbf{N}$ and $z \in E$.

   By **Cauchy Estimates 1.11.1**, for any $z \in E$,

$$|g_n{}'(z)| = \left| \frac{1}{2\pi i} \oint_{\partial D\left(z_0, \frac{r+\rho}{2}\right)} \frac{g_n(\zeta)}{(\zeta - z)^2} d\zeta \right| \leq \frac{1}{2\pi} \frac{M_{D\left(z_0, \frac{r+\rho}{2}\right)}}{\left(\frac{r+\rho}{2} - r\right)^2} \cdot 2\pi \left(\frac{r+\rho}{2} \right).$$

   Thus, if $N_E = M_{D\left(z_0, \frac{r+\rho}{2}\right)} \cdot \frac{r+\rho}{2} \cdot \frac{1}{\left(\frac{1}{2}(\rho - r)\right)^2}$, then $|g_n{}'(z)| \leq N_E$ for all $z \in E$ and all $n \in \mathbf{N}$.

   Consequently,

$$|g_n(\zeta) - g_n(\xi)| \leq \left| \int_\gamma g_n{}'(z) dz \right|,$$

   where γ is the line segment from ζ to ξ. Then,

$$\left| \int_\gamma g_n{}'(z) dz \right| \leq N_E |\zeta - \xi|.$$

   Thus, given $\varepsilon > 0$, we can satisfy the definition of equicontinuity on E by setting $\delta = \frac{\varepsilon}{N_E}$.

   With the claim proven, we have established that (g_n) are equicontinuous on $E = \overline{D(z_0, r)}$, so given $\varepsilon > 0$, there exists $\delta > 0$ so that $|g_\ell(\zeta) - g_\ell(\xi)| < \frac{\varepsilon}{3}$ for all ℓ whenever $\zeta, \xi \in E$ and $|\zeta - \xi| < \delta$.

   By the compactness of E, there is an integer k such that if $z \in E$, then $z \in D(z_j, \delta)$ for some $\{z_1, ..., z_k\} \subseteq E$. If, say, $z \in D(z_j, \delta) \cap E$, then $|g_\ell(z) - g_\ell(z_j)| < \frac{\varepsilon}{3}$ for all ℓ.

   And, since $\{z_1, ..., z_k\}$ is finite, there exists $\alpha = \alpha(E)$ so that $|g_n(z_\ell) - g_m(z_\ell)| < \frac{\varepsilon}{3}$ when $n, m \geq \alpha(E)$ and $z_\ell \in \{z_1, ..., z_k\}$, since we have by construction that $\lim\limits_{m\to\infty} g_m(z_\ell) = w_\ell$ for all ℓ.

   Thus, if $z \in D(z_\ell, \delta) \cap E$, then

$$|g_m(z) - g_n(z)| \leq |g_m(z) = g_m(z_\ell)| + |g_m(z_\ell) - g_n(z_\ell)| + |g_n(z_\ell) - g_n(z)| < \varepsilon,$$

   if $n, m \geq \alpha(E)$.

   Consequently, (g_n) is uniformly Cauchy on E, and (g_n) converges uniformly to some limit function on E, as desired.

Montel's theorem is proven. □

Example 2.2.7. Let $\mathcal{F}$ be $\{z^j\}$, and let $U = D(0,1)$. Then $\mathcal{F}$ is outright bounded by 1, not just on compacta. By **Montel's Theorem 2.2.6**, we are guaranteed the existence of a normally convergent subsequence.

Example 2.2.8. Let $\mathcal{F} = \left\{ \frac{z}{j} \right\}$ on $\mathbf{C}$. Then $\mathcal{F}$ is not bounded on $\mathbf{C}$, but $\mathcal{F}$ is bounded on compacta. **Montel's Theorem 2.2.6** again applies.

Lemma 2.2.9. *Let $U \subseteq \mathbf{C}$ be open. Fix $P \in U$. Let $\mathcal{F} \subseteq H(U, D(0,1))$ be a family with $f(P) = 0$ for all $f \in \mathcal{F}$. Then there is a holomorphic function $f_0 \in H(U, D(0,1))$ that is the normal limit of a sequence $(f_j) \subseteq \mathcal{F}$ such that $|{f_0}'(P)| \geq |f'(P)|$ for all $f \in \mathcal{F}$.*

Proof. We first show that $\sup_{f \in \mathcal{F}} |f'(P)| < \infty$.

Let $f \in H(U, D(0,1))$ with $f(P) = 0$. Choose $r > 0$ so that $\overline{D(P,r)} \subseteq U$. Since Range$(f) \subseteq D(0,1)$, $|f(z)| \leq 1$ for all z. By **Cauchy Estimates 1.11.1**,

$$|f'(P)| = \left| \frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{f(z)}{(z-P)^2} dz \right| \leq \frac{1}{2\pi} \cdot \frac{1}{r^2} \cdot 2\pi r = \frac{1}{r}.$$

Thus, $\frac{1}{r}$ is an upper bound for $|f'(P)|$, and the supremum is finite.

Now, let $\lambda = \sup\{|f'(P)| \mid f \in \mathcal{F}\}$. By the definition of supremum, there is a sequence $(f_j) \subseteq \mathcal{F}$ such that $|{f_j}'(P)| \to \lambda$. Since Range$(f_j) \subseteq D(0,1)$, (f_j) is bounded by 1, and **Montel's Theorem 2.2.6** applies. Consequently, there is a subsequence (f_{j_k}) that converges uniformly on compact sets to a limit function $f_0 \in H(U)$. By **Cauchy Estimates 1.11.1**, the sequence $({f_{j_k}}'(P))$ converges to ${f_0}'(P)$. Therefore, $|{f_0}'(P)| = \lambda$.

All that remains is to check that f_0 maps U to $D(0,1)$; we only know that f_0 maps U to $\overline{D(0,1)}$. However, by the **Maximum Modulus Theorem 1.17.16**, if $f_0(U) \cap \{z \mid |z| = 1\} \neq \emptyset$, then f_0 is a constant of modulus 1. But by hypothesis, $f_0(P) = 0$, so $|f| \not\equiv 1$, and thus $f_0(U) \subseteq D(0,1)$.

(Alternatively, one can argue that by the **Open Mapping Theorem 1.17.7**, since U is open, $f_0(U) \subseteq \overline{D(0,1)}$ is open. Just note that f_0 is not constant. $\square$

Let's develop a few more tools towards the **Riemann Mapping Theorem's 2.2.2** proof. As Wright says, when mathematicians are faced with a hard problem, we'd rather just assign definitions and relabel things, because that's easier than solving the problem. The following definition, holomorphically simply connected sets, is no different; in fact, we will see that in $\mathbf{C}$, holomorphically simply connected is exactly the same as (topologically) simply connected.

Recall what it means for a set to be **holomorphically simply connected 1.16.20**: A connected open set $U \subseteq \mathbf{C}$ is holomorphically simply connected if for each $f \in H(U)$, there is a holomorphic antiderivative; i.e., there exists $F \in H(U)$ such that $F'(z) = f(z)$ for all $z \in U$.

Example 2.2.10. We have seen in **Theorem 1.5.15** that open disks and rectangles are holomorphically simply connected.

Proposition 2.2.11. *If $U_1 \subseteq U_2 \subseteq \cdots$ are holomorphically simply connected, then $\bigcup_{j=1}^{\infty} U_j$ is. (Thus,* $\mathbf{C}$*, a countable union of nested open rectangles/disks, is holomorphically simply connected.*

Proof deferred, since when we show that holomorphic simple connectivity and simple connectivity are the same, it follows from the topological proof.

Thus, we *could* restate the **Riemann Mapping Theorem 2.2.2** in the following, analytic way:

Theorem 2.2.12 (**Riemann Mapping Theorem (Analytic version)**). *Let $U \subseteq \mathbf{C}$ be open, holomorphically simply connected, and such that $U \neq \mathbf{C}$. Then U is conformally equivalent to $D(0,1)$.*

Again, we'll show that all of these properties of a set: conformal equivalence, homeomorphism, etc, which, on their surface, seem like stronger or weaker hypotheses to put on sets, are in fact all the same.

Lemma 2.2.13. *Let $U \subseteq \mathbf{C}$ be holomorphically simply connected. If $f \in H(U)$ and f is nowhere zero, then there exists $h \in H(U)$ such that $e^h \equiv f$ on U.*

This lemma gives a sufficient condition for the existence of holomorphic logarithms. The proof goes about as you could guess:

Proof. Since f is nonvanishing on U, $\frac{f'(z)}{f(z)} \in H(U)$. Since U is holomorphically simply connected, there exists $h \in H(U)$ so that $h'(z) = \frac{f'(z)}{f(z)}$.

Fix $z_0 \in U$. By adding a constant to h (if necessary), we may assume that $e^{h(z_0)} = f(z_0)$. With this normalization, we show that $e^h \equiv f$ on U. To do this, it is enough to show that $g(z) = f(z)e^{-h(z)}$ satisfies $g' \equiv 0$ on U. Since U is connected, and $g(z_0) = 1$, it will follow that $g \equiv 1$ on U, and thus $e^h \equiv f$.

So, see that

$$g'(z) = f'(z)e^{-h(z)} + f(z)\left(-h'(z)e^{-h(z)}\right) = f'(z)e^{-h(z)} - f(z)\frac{f'(z)}{f(z)}e^{-h(z)} = f'(z)e^{-h(z)} - f'(z)e^{-h(z)} = 0$$

on U, as desired. □

Corollary 2.2.14. *If $U \subseteq \mathbf{C}$ is holomorphically simply connected, and $f : U \to \mathbf{C} \setminus \{0\}$ is holomorphic, then there exists a function $g \in H(U)$ that also maps to $\mathbf{C} \setminus \{0\}$ such that $f(z) = (g(z))^2$; i.e., f has a holomorphic square root.*

Proof. Choose $g(z) = e^{\frac{1}{2}h(z)}$, where $h = \frac{f'}{f}$, as in **Lemma 2.2.13**. □

We now approach the proof of the **Riemann Mapping Theorem 2.2.2**. As stated earlier, this is **Part II**, where we construct a one-to-one map from U to $D(0,1)$, and then use the existence of this map to find a biholomorphic function by solving an extremal problem.

Proof of the Riemann Mapping Theorem, Part II. Let $U \subsetneq \mathbf{C}$ be holomorphically simply connected. Fix $P \in U$, and set $\mathcal{F} = \{f \in H(U, D(0,1)) \mid f \text{ is one-to-one}, f(P) = 0\}$.

We will prove the following:

1. $\mathcal{F} \neq \emptyset$.
2. There exists $f_0 \in \mathcal{F}$ such that $|{f_0}'(P)| = \sup_{h \in \mathcal{F}} |h'(P)|$.
3. If $g \in \mathcal{F}$ satisfies $|g'(p)| = \sup_{h \in \mathcal{F}} |h'(P)|$, then g maps U onto $D(0,1)$.

As a note, we prove 1. by direct construction, the proof of 2. is analogous to **Lemma 2.2.9**, and 3. comes via contradiction; if false, we can construct a function $\widehat{g} \in \mathcal{F}$ with $|\widehat{g}'(P)| > |g'(P)|$.

So let us proceed.

1. If U is bounded, then this step is easy. If $a = \frac{1}{2\sup\{|z| | z \in U\}}$ and $b = -aP$, then $f(z) = az + b \in \mathcal{F}$.

   Assume now that U is unbounded. Then since $U \neq \mathbf{C}$, there exists $Q \notin U$. This means that $\varphi(z) = z - Q$ is nonvanishing on U. Since U is holomorphically simply connected, by **Corollary 2.2.14**, there exists $h \in H(U)$ so that $h^2 = \varphi$. Also, h must be one-to-one because φ is one-to-one. Even stronger, though, it cannot be that there exist $z_1, z_2 \in U$ such that $h(z_1) = -h(z_2)$, because this would imply that $\varphi(z_1) = \varphi(z_2)$.

   So h is a nonconstant holomorphic mapping, and therefore by the **Open Mapping Theorem 1.17.7**, an open mapping. Thus, the image of h contains a disk, $D(b,r)$, for some $b \in \mathbf{C}$ and $r > 0$. This means that $\operatorname{Range} h \cap D(-b,r)$ can be made to be empty; take $r < \frac{1}{2}|b|$.

   We therefore define

   $$f(z) = \frac{r}{2(h(z)+b)}.$$

   Since $|h(z) - (-b)| > r$ for all $z \in U$, $f : U \to D(0,1)$. The function h is one-to-one, so f is as well. Finally, it may be the case that $f(P) \neq 0$, but composing f with an appropriate Möbius transformation will yield $\widetilde{f} \in \mathcal{F}$.

2. Here, **Lemma 2.2.9** will yield the desired result, once we conclude that the derivative maximizer is itself one-to-one, since that is the only condition to be in $\mathcal{F}$ not met by the lemma.

   We may suppose that $(f_j) \subseteq \mathcal{F}$ converges normally to f_0 on U, and $|{f_0}'(P)| = \sup_{f\in\mathcal{F}}|f'(P)|$.

   We will show that f_0 is one-to-one in $D(0,1)$ using the **argument principle 1.17.4**, and **Hurwitz's Theorem 1.17.14** in particular.

   Fix $b \in U$. Let $g_j(z) = f_j(z) - f_j(b)$, and consider (g_j) on $U \setminus \{b\}$. Also, we recall **Hurwitz 1.17.14**; click the link for the full theorem and proof.

   > **Hurwitz's Theorem:** If (h_j) are nonvanishing and $h_j \to h$ normally on V, then $h \equiv 0$ or h is nonvanishing on V.

   Consequently, $f_0(z) - f_0(b)$ is identically 0, or nowhere vanishing, on $U \setminus \{b\}$.

   Next, for any $h \in \mathcal{F}$, it must hold that $h'(P) \neq 0$, because otherwise h would fail to be one-to-one. And since $\mathcal{F} \neq \emptyset$ by 1., it follows that $\sup_{h\in\mathcal{F}}|h'(P)| > 0$.

   The function f_0 satisfies $|{f_0}'(P)| = \sup_{h\in\mathcal{F}}|h'(P)|$; hence, ${f_0}'(P) \neq 0$, so f_0 is not identically zero. Therefore, $f_0(z) - f_0(b)$ is nowhere zero on $U \setminus \{b\}$. Since this statement holds for all $b \in U$, it follows that f_0 is one-to-one, as desired. Now, **Lemma 2.2.9** gives us the rest.

3. Let $g \in \mathcal{F}$ be a derivative maximizer, and suppose there exists $R \in D(0,1)$ such that $R \notin \operatorname{Range} g$. We'll show this leads to a contradiction, and therefore g is onto.

   Set $\varphi(z) = \frac{g(z)-R}{1-\overline{R}g(z)}$. Note that φ is nonvanishing, and $\varphi : U \to D(0,1)$. Since U is holomorphically simply connected, there exists $\psi \in H(U, D(0,1))$ with $\psi^2 = \varphi$, by **Corollary 2.2.14**. It follows that ψ is one-to-one, as g and hence φ are.

   However, since φ is nonvanishing, $\varphi \notin \mathcal{F}$. We can fix this with another Möbius transformation; set $\rho(z) = \frac{\psi(z)-\psi(P)}{1-\overline{\psi(P)}\psi(z)}$. Then $\rho(P) = 0$, $\rho \in H(U, D(0,1))$, and ρ is one-to-one. Therefore, $\rho \in \mathcal{F}$. We will use ρ to reach our contradiction; we will see that $|\rho'(P)| > |g'(P)|$.

   To that end, see that

$$\begin{aligned}\rho'(P) &= \frac{\left(1-|\psi(P)|^2\right)\psi'(P) - (\psi(P)-\psi(P))\left(-\psi'(P)\overline{\psi(P)}\right)}{\left(1-|\psi(P)|^2\right)^2}\\ &= \frac{\psi'(P)}{1-|\psi(P)|^2}.\end{aligned}$$

   Now recall that $\varphi = \psi^2$, so

$$2\psi(P)\psi'(P) = \varphi'(P) = \frac{\left(1-g(P)\overline{R}\right)g'(P) - (g(P)-R)\left(-g'(P)\overline{R}\right)}{\left(1-g(P)\overline{R}\right)^2}.$$

   Since $g \in \mathcal{F}$, $g(P) = 0$, so

$$2\psi(P)\psi'(P) = \varphi'(P) = g'(P) - g'(P)|R|^2 = g'(P)\left(1-|R|^2\right).$$

   Consequently,

$$\begin{aligned}\rho'(P) &= \frac{1}{1-|\psi(P)^2|}\cdot\frac{1-|R|^2}{2\psi(P)}\cdot g'(P)\\ &= \frac{1}{1-|\varphi(P)|}\cdot\frac{1-|R|^2}{2\psi(P)}\cdot g'(P)\\ &= \frac{1}{1-|R|}\cdot\frac{1-|R|^2}{2\psi(P)}\cdot g'(P)\\ &= \frac{1+|R|}{2\psi(P)}\cdot g'(P).\end{aligned}$$

We know that $1 + |R| > 1$, and $|\psi(P)| = \sqrt{|R|}$. Thus,

$$\frac{1 + |R|}{2\sqrt{|R|}} > 1,$$

and therefore,

$$|\rho'(P)| > |g'(P)|.$$

Here is the contradiction we seek.

The second part of the Riemann Mapping Theorem is thus proven. □

2.3 Harmonic Functions

Definitions: harmonic conjugate, small circle mean value property
Main Idea: We return to harmonic functions in much greater detail here. There are plenty of good results, many of which come from results involving holomorphic functions. Harmonic functions are smooth, satisfy a maximum/minimum principle, a mean value property, and other results. We also discover the Poisson Integral Formula, the Schwarz Reflection Principle, and Harnack's Principle.

Recall what it means for a function u to be **harmonic 1.5.10**, that $u \in C^2(U)$ and $\Delta u = 0$. We write $h(U)$ for the set of harmonic functions on U, like we write $H(U)$ for the set of holomorphic functions on U.
Recall also that if $U \subseteq \mathbf{C}$ is open and $F = u + iv \in H(U)$, then both u and v are harmonic.
We now discover that if $F_1 = u_1 + iv_1$ and $F_2 = u_2 + iv_2$ are holomorphic on U with $u_1 = u_2$, then $F_1 - F_2 = i(v_1 - v_2)$ clearly. But $F_1 - F_2 \in H(U)$. If $F_1 - F_2$ is nonconstant, then $\mathrm{Range}(F_1 - F_2)$ is open, by the **Open Mapping Theorem 1.17.7**. But $\mathrm{Range}(F_1 - F_2) = \mathrm{Range}(i(v_1 - v_2)) \subseteq \{0\} \times i\mathbf{R}$, which contains no open sets in $\mathbf{C}$. Thus, $F_1 - F_2 = ic$ for some $c \in \mathbf{R}$. From this, we can conclude that u_1 carries all of the essential information of F_1.
Also, note that if $u \in h(U)$, then $\operatorname{Re} u, \operatorname{Im} u \in h(U)$. Thus, we generally just assume that harmonic functions are real valued.
Finally, recall **Corollary 1.5.14**; if $u \in h(D(z, r))$ for $z \in \mathbf{C}$ and some $r > 0$, then there exists a function $F \in H(D(z, r))$ so that $u = \operatorname{Re} F$. Note that **Corollary 1.5.14** was proven for disks and rectangles, but indeed holds for any holomorphically simply connected open sets.

Definition 2.3.1. If $F = u + iv \in H(U)$, then v is called a **harmonic conjugate** of u, or we say that u and v are harmonic conjugates.

Note that by above, harmonic conjugates aren't unique, but they differ by at most a constant.
Let's now explore harmonic functions in much more detail.

Lemma 2.3.2. *If $u \in h(U)$, then $u \in C^\infty(U)$.*

Proof. It is enough to check that $u \in C^\infty(D)$, where D is a disk. By **Corollary 1.5.14**, there exists a function $F \in H(D)$ such that $F = u + iv$ on D. As F is holomorphic, $F \in C^\infty(D)$. Consequently, $u = \operatorname{Re} F \in C^\infty(D)$. □

Lemma 2.3.3. *If U is a holomorphically simply connected open set, and $u \in h(U)$, then u has a harmonic conjugate v on U.*

Proof. Define H by $H(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$. Since $\Delta u = 0$,

$$\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right] = -\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial y}\right] \qquad \text{and} \qquad \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x}\right] = -\frac{\partial}{\partial x}\left[-\frac{\partial u}{\partial y}\right],$$

since $u \in C^2(U)$. Hence, H satisfies the **Cauchy-Riemann equations 1.5.4** and is C^1; therefore we get that $H \in H(U)$.

Since U is holomorphically simply connected, there exists $F \in H(U)$ so that $F'(z) = H(z)$. Write $F = \widetilde{u} + i\widetilde{v}$. Then

$$F' = \frac{\partial \widetilde{u}}{\partial x} + i\frac{\partial \widetilde{v}}{\partial x} = \frac{\partial \widetilde{u}}{\partial x} - i\frac{\partial \widetilde{u}}{\partial y},$$

where the second equality holds by **Cauchy-Riemann 1.5.4**. Now, since $F' = H = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$, $\frac{\partial \widetilde{u}}{\partial x} = \frac{\partial u}{\partial x}$ and $\frac{\partial \widetilde{u}}{\partial y} = \frac{\partial u}{\partial y}$. Thus, $u - \widetilde{u}$ is identically constant; say $u - \widetilde{u} \equiv c$.

Then $F - c = u + i\widetilde{v}$ is holomorphic on U, and $\mathrm{Re}(F - c) = u$, as required. ☐

Now, harmonic functions and holomorphic functions share many properties. Here is the first:

Theorem 2.3.4 (**Maximum Principle**). *Let $U \subseteq \mathbf{C}$ be a connected, open set, and let $u \in h(U)$. If there is a point $P_0 \in U$ at which $u(P_0) = \sup_{Q \in U} u(Q)$, then u is constant on U.*

Proof. Let

$$M = \left\{ P \in U \mid u(P) = \sup_{\zeta \in U} u(\zeta) \right\}.$$

Assume $M \neq \emptyset$. We will show that M is both open and closed in U. Since U is connected, $M = U$, and as M is this preimage construction, the result will be proven.

To see that M is closed, simply note that since $u \in C(U)$ and $\left\{ \sup_{\zeta \in U} u(\zeta) \right\}$ is a closed singleton,

$$M = u^{-1}\left(\left\{ \sup_{\zeta \in U} u(\zeta) \right\} \right)$$

is closed.

To see that M is open, suppose $P \in M$. Choose $r > 0$ small enough that $D(P, r) \subseteq U$. On $D(P, r)$, there exists $H \in H(D(P, r))$ such that $u = \mathrm{Re}\, H$, by **Corollary 1.5.14**. Set $F(z) = e^{H(z)}$. Then

$$|F(P)| = \left| e^{H(P)} \right| = \left| e^{u(P) + iv(P)} \right| = e^{u(P)} = \sup_{\zeta \in D(P,r)} |F(\zeta)|.$$

By the **Maximum Modulus Principle 1.17.15** (for holomorphic functions), F must be constant on $D(P, r)$. Consequently, H, and therefore u, are constant on $D(P, r)$ as well. Thus, M is open, and the proof is concluded. ☐

Corollary 2.3.5 (**Minimum Principle**). *If $U \subseteq \mathbf{C}$ is a connected, open set, $u \in h(U)$, and there exists $P_0 \in U$ at which $u(P_0) = \inf_{Q \in U} u(Q)$, then u is constant on U.*

Proof. Apply the **Maximum Principle 2.3.4** to $-u$. ☐

Corollary 2.3.6 (**Maximum Theorem**). *Let $U \subseteq \mathbf{C}$ be a bounded, connected, open set. If $u \in h(U) \cap C(\overline{U})$, then*

$$\max_{\overline{U}} u = \max_{\partial U} u \qquad \text{and} \qquad \min_{\overline{U}} u = \min_{\partial U} u.$$

Proof. Since $\overline{U}$ is closed and bounded, it is compact. Since $u \in C(\overline{U})$, u attains its maximum, say at $P \in \overline{U}$. If $P \in \partial U$, then there is nothing to prove. If $P \in U$, then the **Maximum Principle 2.3.4** implies $u \equiv u(P)$, and we are again done.

For the minimum statement, run the argument for $-u$. ☐

Another property held by harmonic functions that is similar to holomorphic functions is the following, analogous to the **Cauchy Integral Formula 1.9.3**:

Theorem 2.3.7 (**Mean Value Property**). *Suppose $U \subseteq \mathbf{C}$ is open, and $u \in h(U)$. If $P \in U$ and $D(P,r) \subseteq U$, then*

$$u(P) = \frac{1}{2\pi}\int_0^{2\pi} u\left(P + re^{i\theta}\right) d\theta.$$

Proof. Choose $s > r$ such that $D(P,s) \subseteq U$. Let $H \in H(D(P,s))$ such that $\operatorname{Re} H = u$. Let $\gamma = P + re^{it}$, $t \in [0, 2\pi]$.

By the **Cauchy Integral Formula 1.9.3**,

$$\begin{aligned}
u(P) + iv(P) = H(P) &= \frac{1}{2\pi i}\oint_\gamma \frac{H(z)}{z-P} dz \\
&= \frac{1}{2\pi i}\int_0^{2\pi} \frac{H\left(P+re^{it}\right)}{P+re^{it}-P} ire^{it} dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} H\left(P+re^{it}\right) dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} u\left(P+re^{it}\right) dt + \frac{i}{2\pi}\int_0^{2\pi} v\left(P+re^{it}\right) dt.
\end{aligned}$$

Taking the real part yields the desired result. $\square$

Note that if $u_1, u_2 \in h(D(0,1)) \cap C(\overline{D(0,1)})$ and $u_1 = u_2$ on $\partial D(0,1)$, then $u_1 \equiv u_2$ on $\overline{D(0,1)}$. This follows from the **Maximum Theorem 2.3.6** applied to $u_1 - u_2$. The **Mean Value Property 2.3.7** would only yield $u_1(0) = u_2(0)$ in this case.

We now turn to the discussion of the Poisson Integral. Before we do so, recall some salient facts about **Möbius transformations 2.1.5**:

1. For $a \in D(0,1)$, $\varphi_a : D(0,1) \to D(0,1)$, $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$ is a conformal map (**Lemma 2.1.6**);
2. $(\varphi_a)^{-1} = \varphi_{-a}$ (in the proof of **Schwarz-Pick 1.18.2**);
3. $\varphi_a(0) = -a$ (obvious computation).

Also note the following:

Lemma 2.3.8. *If $u \in h(U)$ and $H \in H(V,U)$, then $u \circ H \in h(V)$.*

Proof. Recall from **Lemma 1.5.12** that $\Delta = 4\frac{\partial^2}{\partial z \partial \bar{z}}$.

We can therefore conclude that

$$\begin{aligned}
\Delta\big[u \circ H\big] &= 4\frac{\partial}{\partial z}\left[\frac{\partial}{\partial \bar{z}}\big[u \circ H\big]\right] \\
&= 4\frac{\partial}{\partial z}\left[\frac{\partial u}{\partial z}\cdot\frac{\partial H}{\partial \bar{z}} + \frac{\partial u}{\partial \bar{z}}\cdot\frac{\partial \overline{H}}{\partial \bar{z}}\right] \\
&= 4\frac{\partial}{\partial z}\left[\left.\frac{\partial u}{\partial \bar{z}}\right|_{H(z)}\cdot\frac{\partial \overline{H}}{\partial \bar{z}}\right] \\
&= 4\frac{\partial^2 u}{\partial z \partial \bar{z}}\cdot\frac{\partial H}{\partial z}\cdot\frac{\partial \overline{H}}{\partial \bar{z}} \\
&= \Delta\left. u\right|_{H(z)}\cdot\frac{\partial H}{\partial z}\cdot\frac{\partial \overline{H}}{\partial \bar{z}} \\
&= 0,
\end{aligned}$$

as $u \in h(U)$. Thus $u \circ H \in h(V)$. $\square$

Note that if $H \notin H(V,U)$, then the above is not generally true. We need the strength of holomorphicity, namely the fact that $\frac{\partial H}{\partial \bar{z}} = 0$.

Theorem 2.3.9 (**The Poisson Integral Formula).** *Let $u : U \to \mathbf{R}$ be a harmonic function on an open set U containing $D(0,1)$. If $a \in D(0,1)$, then*

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u\left(e^{it}\right)\frac{1-|a|^2}{\left|a-e^{it}\right|^2}dt.$$

Note as a preliminary that the expression $\frac{1-|a|^2}{2\pi|a-e^{it}|^2}$ is called the Poisson kernel[8] for the unit disk. If we write $a = re^{i\theta}$, then we can also restate the **Poisson Integral Formula 2.3.9** as

$$u\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_0^{2\pi} u\left(e^{it}\right)\frac{1-r^2}{1-2r\cos(\theta-t)+r^2}dt.$$

The Poisson kernel is $P_r(\theta - t) = \frac{1-r^2}{2\pi(1-2r\cos(\theta-t)+r^2)}$, with, concisely,

$$u\left(re^{i\theta}\right) = \int_0^{2\pi} u\left(e^{it}\right)P_r(\theta-t)dt.$$

Notice the convolution in t and $\theta - t$. See PDEs for further discussion.

Let's now prove the **Poisson Integral Formula 2.3.9**.

Proof. We apply the **Mean Value Property 2.3.7** to the harmonic function $u \circ \varphi_{-a}$ (by **Lemma 2.3.8**). See that

$$\begin{aligned} u(a) = u\circ\varphi_{-a}(0) &= \frac{1}{2\pi}\int_0^{2\pi} u\left(\varphi_{-a}\left(e^{it}\right)\right)dt \\ &= \frac{1}{2\pi i}\int_0^{2\pi}\frac{u\left(\varphi_{-a}\left(e^{it}\right)\right)}{e^{it}}ie^{it}dt \\ &= \frac{1}{2\pi i}\oint_{\partial D(0,1)}\frac{u\left(\varphi_{-a}(\zeta)\right)}{\zeta}d\zeta. \end{aligned}$$

Now, let $\zeta = \varphi_a(\xi)$. This is a one-to-one C^1 transformation on a neighborhood of $\overline{D(0,1)}$ (by notes we made about Möbius transformations above), hence on $\partial D(0,1)$ in particular. Also, we know that $\varphi_a(\partial D(0,1)) = \partial D(0,1)$.

Next,

$$\varphi_a{}'(\xi) = \frac{1-|a|^2}{(1-\overline{a}\xi)^2}.$$

Thus,

$$\begin{aligned} u(a) &= \frac{1}{2\pi i}\oint_{\partial D(0,1)}\frac{u\left(\varphi_{-a}(\zeta)\right)}{\zeta}d\zeta \\ &= \frac{1}{2\pi i}\oint_{\partial D(0,1)}\frac{u(\xi)}{\varphi_a(\xi)}\varphi_a{}'(\xi)d\xi \\ &= \frac{1}{2\pi i}\int_0^{2\pi}\frac{u\left(e^{it}\right)}{\frac{e^{it}-a}{1-\overline{a}e^{it}}}\frac{1-|a|^2}{\left(1-\overline{a}e^{it}\right)^2}ie^{it}dt \\ &= \frac{1}{2\pi}\int_0^{2\pi}u\left(e^{it}\right)\frac{1-|a|^2}{\left|e^{it}-a\right|^2}dt. \end{aligned}$$

The Poisson Integral Formula is thus proven. ∎

[8] As far as Dr. Harrington knows, nothing to do with kernels in algebra.

We remark to draw a distinction between holomorphic and harmonic functions that the **Cauchy Integral Formula 1.9.3** both reproduces and produces holomorphic functions. For example, if $f \in C(\partial D(0,1))$, then

$$F(z) = \frac{1}{2\pi i} \oint_{\partial D(0,1)} \frac{f(\zeta)}{\zeta - z} d\zeta \in H(D(0,1)),$$

but, and this is critical to note, there may be little or no relationship between f and F. We saw that, for instance, if $f(\zeta) = \frac{1}{\zeta}$, then $f(\zeta) = \overline{\zeta}$ on $\partial D(0,1)$, and $F \equiv 0$.

The situation is different for harmonic functions; there is a nice relationship between the functions involved in the **Poisson Integral Formula 2.3.9**.

Theorem 2.3.10 (**A solution of the Dirichlet Problem on** $D(0,1)$)**.** *Let* $f \in C(\partial D(0,1))$. *Define*

$$u(z) = \begin{cases} \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} f\left(e^{it}\right) \frac{1-|z|^2}{\left|z - e^{it}\right|^2} dt & \text{for } z \in D(0,1); \\ f(z) & \text{for } z \in \partial D(0,1). \end{cases}$$

Then $u \in h(D(0,1)) \cap C(\overline{D(0,1)})$.

The Dirichlet problem asks us exactly what the theorem provides; given a function that is continuous on the boundary of some open set, can we produce a function that agrees continuously with the given one and is harmonic inside? Thus this theorem says "yes, and here's how," if your open set is $D(0,1)$.

Proof. We first will show that $u \in h(D(0,1))$. Via partial fractions, we may write

$$\frac{1-|z|^2}{\left|z-e^{it}\right|^2} = \frac{e^{it}}{e^{it}-z} + \frac{e^{-it}}{e^{-it}-\overline{z}} - 1.$$

Now, for $z \in D(0,1)$, we have by construction that

$$\begin{aligned} u(z) &= \frac{1}{2\pi}\int_0^{2\pi} f\left(e^{it}\right) \frac{1-|z|^2}{\left|z-e^{it}\right|^2} dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} f\left(e^{it}\right) \frac{e^{it}}{e^{it}-z} dt + \frac{1}{2\pi}\int_0^{2\pi} f\left(e^{it}\right) \frac{e^{-it}}{e^{-it}-\overline{z}} dt - \frac{1}{2\pi}\int_0^{2\pi} f\left(e^{it}\right) dt. \end{aligned}$$

The first integral produces a holomorphic function. The second integral produces an antiholomorphic function (one "holomorphic" in $\overline{z}$). The third integral is constant in z and $\overline{z}$.

Since we know that $\Delta = 4\frac{\partial^2}{\partial z \partial \overline{z}}$, both holomorphic and antiholomorphic functions are harmonic. Further, constant functions are clearly harmonic as well. Therefore, $u \in h(D(0,1))$, as we wanted to see.

Next, we need to show that $u \in C(\overline{D(0,1)})$. The idea is that the **Poisson Integral Formula 2.3.9** can be thought of as a weighted average of the values of f on $\partial D(0,1)$ with the weight $P_r(\theta - t) = \frac{1-r^2}{2\pi(1-2r\cos(\theta-t)+r^2)}$, which is positive and has mass 1. Also, we'll see that if $t \neq \theta$, $\lim\limits_{r \to 1^-} P_r(\theta - t) = 0$.

Observe that the function $v(z) \equiv 1$ is harmonic near $\overline{D(0,1)}$, and by the **Poisson Integral Formula 2.3.9**,

$$1 = v(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{\left|z-e^{it}\right|^2} dt,$$

so indeed the Poisson kernel has mass 1.

If $P_0 = e^{i\theta_0} \in \partial D(0,1)$ is fixed, and $z = re^{i\theta} \in D(0,1)$ is near P_0, then

$$\begin{aligned} |u(P_0) - u(z)| &= \left| u\left(e^{i\theta_0}\right) - \frac{1}{2\pi}\int_0^{2\pi} f\left(e^{it}\right) \frac{1-r^2}{\left|re^{i\theta}-e^{it}\right|^2} dt \right| \\ &= \left| f\left(e^{i\theta_0}\right) - \frac{1}{2\pi}\int_0^{2\pi} f\left(e^{it}\right) \frac{1-r^2}{\left|re^{i\theta}-e^{it}\right|^2} dt \right| \\ &= \left| \frac{1}{2\pi}\int_0^{2\pi} \left(f\left(e^{i\theta_0}\right) - f\left(e^{it}\right)\right) \frac{1-r^2}{\left|1-re^{i(\theta-t)}\right|^2} dt \right|. \end{aligned}$$

Now, $f\left(e^{i\theta_0}\right)-f\left(e^{it}\right)$ is small if t is near θ_0, and $\frac{1-r^2}{\left|1-re^{i(\theta-t)}\right|^2}$ is small if r is near 1 and $\theta-t$ is away from 0. Since $\partial D(0,1)$ is compact and $f\in C(\partial D(0,1)$, f is uniformly continuous on $\partial D(0,1)$. Let $M=\max_{\partial D(0,1)}|f|$, and let $\varepsilon>0$. The uniform continuity of f means that there exists $\delta>0$ such that if $|s-t|<\delta$, then $\left|f\left(e^{is}\right)-f\left(e^{it}\right)\right|<\frac{\varepsilon}{2}$. Choose $z\in D(0,1)$ such that $z=re^{i\theta}$ satisfies:

1. $|\theta-\theta_0|<\frac{\delta}{3}$,
2. $r\geq\frac{1}{2}$, and
3. $|1-r|<\frac{\delta^2\varepsilon}{1000M}$.

Then

$$\begin{aligned}|u(P_0)-u(z)|&\leq\frac{1}{2\pi}\int_{\{t||t-\theta_0|<\delta\}}\left|f\left(e^{it}\right)-f\left(e^{i\theta}\right)\right|\frac{1-r^2}{\left|1-re^{i(\theta-t)}\right|^2}dt\\&\quad+\frac{1}{2\pi}\int_{\{t||t-\theta_0|\geq\delta\}}\left|f\left(e^{i\theta_0}\right)-f\left(e^{it}\right)\right|\frac{1-r^2}{\left|1-re^{i(\theta-t)}\right|^2}dt.\end{aligned}$$

We analyze the two pieces separately. See first that

$$\begin{aligned}\frac{1}{2\pi}\int_{\{t||t-\theta_0|<\delta\}}\left|f\left(e^{it}\right)-f\left(e^{i\theta}\right)\right|\frac{1-r^2}{\left|1-re^{i(\theta-t)}\right|^2}dt&\leq\frac{1}{2\pi}\int_{\{t||t-\theta_0|<\delta\}}\frac{\varepsilon}{2}\cdot\frac{1-r^2}{\left|1-re^{i(\theta-t)}\right|^2}dt\\&<\frac{\varepsilon}{2}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{\left|1-re^{i(\theta-t)}\right|^2}dt\right)=\frac{\varepsilon}{2}.\end{aligned}$$

To estimate the second piece, first note that for $0\leq|\alpha|\leq\pi$, $1-\cos\alpha\geq\frac{\alpha^2}{20}$. (This could be shown via, e.g., a Taylor series.) Then, by our choice of δ and r, we see that

$$\begin{aligned}\left|1-re^{i(\theta-t)}\right|^2&=\left(1-re^{i(\theta-t)}\right)\left(1-re^{-i(\theta-t)}\right)\\&=1-2\operatorname{Re}\left(re^{i(\theta-t)}\right)+r^2\\&=1-2r\cos(\theta-t)+r^2\\&=1-2r\cos(\theta-t)+r^2-2r+2r\\&=(1-r)^2+2r(1-\cos(\theta-t))\\&\geq 2r(1-\cos(\theta-t))\\&\geq\frac{r}{10}(\theta-t)^2.\end{aligned}$$

Therefore, we can bound the second piece as follows:

$$\frac{1}{2\pi}\int_{\{t||t-\theta_0|\geq\delta\}}\left|f\left(e^{i\theta_0}\right)-f\left(e^{it}\right)\right|\frac{1-r^2}{\left|1-re^{i(\theta-t)}\right|^2}dt\leq\frac{1}{2\pi}\int_{\{t||t-\theta_0|\geq\delta\}}20M\frac{1-r^2}{r(\theta-t)^2}dt.$$

Now, we know that $|\theta-\theta_0|<\frac{\delta}{3}$ and $|t-\theta_0|\geq\delta$, so $|\theta-t|\geq\frac{2\delta}{3}$. Therefore,

$$\frac{1}{2\pi}\int_{\{t||t-\theta_0|\geq\delta\}}20M\frac{1-r^2}{r(\theta-t)^2}dt\leq\frac{1}{2\pi}\int_0^{2\pi}20M\frac{(1-r)(1+r)}{r\left(\frac{2}{3}\delta\right)^2}dt<\frac{\varepsilon}{2}.$$

In conclusion, $|u(P_0)-u(z)|\ \underline{<\varepsilon,\text{ so}}$ we have continuity on $\partial D(0,1)$. As u is harmonic inside $D(0,1)$, we now have continuity on all of $\overline{D(0,1)}$, as desired. The theorem is proven. $\square$

We'll now explore regularity of harmonic functions. We will show that continuous functions that satisfy the **Mean Value Property 2.3.7** are actually harmonic, and hence C^∞.

Definition 2.3.11. Let $U \subseteq \mathbf{C}$ be open and let $h \in C(U)$. We say that h has the **small circle mean value property** if for each $P \in U$, there exists $\varepsilon = \varepsilon_P > 0$ such that $\overline{D(P, \varepsilon_P)} \subseteq U$ and for every $0 < \varepsilon < \varepsilon_P$, h satisfies the **Mean Value Property 2.3.7**; i.e.,

$$h(P) = \frac{1}{2\pi} \int_0^{2\pi} h\left(P + \varepsilon e^{i\theta}\right) d\theta.$$

Note that clearly, ε_P may vary with P.

Note that if u is a harmonic function, then u satisfies the small circle mean value property, by the **Mean Value Property 2.3.7**. Our goal is to show the other direction, that continuous functions satsifying the small circle mean value property are harmonic. We do in **Theorem 2.3.13**.

Lemma 2.3.12. *Let $V \subseteq \mathbf{C}$ be a connected, open set. Let $g \in C(V)$. If g has the small circle mean value property and there exists $P_0 \in V$ such that $g(P_0) = \sup_{Q \in V} g(Q)$, then g is constant on V.*

Proof. As to be expected in all proofs so far of this manner, we're going to show a nonempty preimage of g is both open and closed, hence the whole set, as U is connected.

Set $s = \sup_{Q \in V} g(Q)$, and let $M = \{z \in V \mid g(z) = s\}$. Now, $M \neq \emptyset$ because $P_0 \in M$. M is closed, because $M = g^{-1}(\{s\})$, g is continuous, and $\{s\}$ is closed. To see M is open, let $P \in M$ and choose ε_P as in the definition of the **small circle mean value property 2.3.11**. Then, for $0 < \varepsilon < \varepsilon_P$,

$$s = g(P) = \frac{1}{2\pi} \int_0^{2\pi} g\left(P + \varepsilon e^{i\theta}\right) d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} s d\theta = s.$$

Thus, the inequality must be an equality. Since g is continuous and bounded from above by s, $g\left(P + \varepsilon e^{i\theta}\right) = s$ for all $\theta \in [0, 2\pi]$ and $0 \leq \varepsilon < \varepsilon_P$, by continuity. Hence, $D(P, \varepsilon_P) \subseteq M$, so M is open.

Therefore, $M = V$, as we wished to show. ☐

Theorem 2.3.13. *If $U \subseteq \mathbf{C}$ is open, and $h \in C(U)$ satisfies the small circle mean value property, then $h \in h(U)$.*

Proof. Let $D \subseteq U$ be an open disk such that $\overline{D} \subseteq U$. By translation and dilation and application of **Theorem 2.3.10**, there exists $u_D \in h(D, \mathbf{R})$ such that $\widehat{u_D}$ is defined on $\overline{D}$ by

$$\widehat{u_D}(z) = \begin{cases} u_D(z) & \text{for } z \in D; \\ h(z) & \text{for } z \in \partial D. \end{cases}$$

We know also that $\widehat{u_D}$ is continuous in $\overline{D}$.

We want to show that $h = u_D$ on D. Let $w = h - \widehat{u_D}$ on $\overline{D}$. Then $w \equiv 0$ on ∂D, and w satisfies the small circle mean value property, since both h and $\widehat{u_D}$ do. From **Lemma 2.3.12**, $w \leq 0$ on D, for otherwise, $\sup w > 0$ and would be attained inside D, forcing w to be a positive constant. We can apply the same reasoning to $-w$, which establishes that $-w \leq 0$ on D. Hence, $w \equiv 0$ on D, and $h \in h(D)$. Since $D \subseteq U$ was arbitrary, $h \in h(U)$. ☐

Corollary 2.3.14. *If (h_j) is a sequence of real valued harmonic functions (or, equivalently by **Theorem 2.3.13**, a sequence of real valued functions that satisfy the small circle mean value property) that converge uniformly on compact sets of U (normally) to $h : U \to \mathbf{R}$, then $h \in h(U)$.*

Proof. Since $(h_j) \to h$ uniformly on compacta, $h \in C(U)$. If $\overline{D(P, r)} \subseteq U$, then

$$h_j(P) = \frac{1}{2\pi} \int_0^{2\pi} h_j\left(P + re^{i\theta}\right) d\theta$$

for each $j \in \mathbf{N}$. By uniform convergence, we may commute the limit and the integral, so therefore

$$h(P) = \frac{1}{2\pi} \int_0^{2\pi} h\left(P + re^{i\theta}\right) d\theta.$$

Thus h has the small circle mean value property, and by **Theorem 2.3.13**, h is harmonic, as desired. ☐

We're now going to focus on a remarkable property of harmonic functions, the **Schwarz Reflection Principle 2.3.17**. First, a lemma:

Lemma 2.3.15. *Let $V \subseteq \mathbf{C}$ be a connected, open set. Suppose that $V \cap \mathbf{R} = \{x \in \mathbf{R} \mid a < x < b\}$. Set $U = \{z \in V \mid \operatorname{Im} z > 0\}$. Assume that $v : U \to \mathbf{R}$ is harmonic, and that for each $\zeta \in V \cap \mathbf{R}$,*

$$\lim_{\substack{z \to \zeta \\ z \in U}} v(z) = 0.$$

Set $\widehat{U} = \{\overline{z} \mid z \in U\}$. Define

$$\widehat{v}(z) = \begin{cases} v(z) & \text{if } z \in U; \\ 0 & \text{if } z \in V \cap \mathbf{R}; \\ -v(\overline{z}) & \text{if } z \in \widehat{U}. \end{cases}$$

Then $\widehat{v} \in h\left(U \cup \widehat{U} \cup \{x \in \mathbf{R} \mid a < x < b\}\right)$.

Proof. Certainly, $\widehat{v}$ is continuous on $W = U \cup \widehat{U} \cup \{x \in \mathbf{R} \mid a < x < b\}$. We will show that $\widehat{v}$ satisfies the small circle mean value property on W, and thus by **Theorem 2.3.13**, $\widehat{v}$ is harmonic on W.

If $P \in U$, then $\widehat{v}$ satisfies the small circle mean value property, since $\widehat{v} = v$ on a neighborhood of P, and $v \in h(U)$. So $\widehat{v} \in h(U)$.

If $P \in \widehat{U}$, then $\widehat{v}(z) = -v(\overline{z}) : \widehat{U} \to \mathbf{R}$ is also harmonic, via the following straightforward computation:

$$\Delta\left[\widehat{v}\right] = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) v(x,-y) = \frac{\partial^2 v}{\partial x^2}(x,-y) + \frac{\partial^2 v}{\partial y^2}(x,-y) = 0.$$

Thus $\widehat{v} \in h(\widehat{(U)})$.

Finally, we have left to show that the small circle mean value property holds for an arbitrary point $P \in \{x \in \mathbf{R} \mid a < x < b\}$. So let $P \in \{x \in \mathbf{R} \mid a < x < b\}$ and choose ε_P so that $\overline{D(P, \varepsilon_P)} \subseteq W$. We know that $v(P) = 0$, and we let $0 < \varepsilon < \varepsilon_P$. Now,

$$\begin{aligned} \int_0^{2\pi} \widehat{v}\left(P + \varepsilon e^{i\theta}\right) d\theta &= \int_0^{\pi} \widehat{v}\left(P + \varepsilon e^{i\theta}\right) d\theta + \int_{\pi}^{2\pi} \widehat{v}\left(P + \varepsilon e^{i\theta}\right) d\theta \\ &= \int_0^{\pi} v\left(P + \varepsilon e^{i\theta}\right) d\theta + \int_0^{\pi} \widehat{v}\left(P + \varepsilon e^{i(\pi+\theta)}\right) d\theta \\ &= \int_0^{\pi} v\left(P + \varepsilon e^{i\theta}\right) d\theta - \int_0^{\pi} v\left(P + \varepsilon e^{-i(\pi+\theta)}\right) d\theta \\ &= 0, \end{aligned}$$

since via the substitution $\alpha = \pi - \theta$,

$$\begin{aligned} \int_0^{\pi} v\left(P + \varepsilon e^{-i(\pi+\theta)}\right) d\theta &= \int_0^{\pi} v\left(P + \varepsilon e^{-i\pi} e^{-i\theta}\right) d\theta \\ &= \int_0^{\pi} v\left(P + \varepsilon(-1) e^{-i\theta}\right) d\theta \\ &= \int_0^{\pi} v\left(P + \varepsilon e^{i\pi} e^{-i\theta}\right) d\theta \\ &= \int_0^{\pi} v\left(P + \varepsilon e^{i(\pi-\theta)}\right) d\theta \\ &= -\int_{\pi}^{0} v\left(P + \varepsilon e^{i\alpha}\right) d\alpha \\ &= \int_0^{\pi} v\left(P + \varepsilon e^{i\alpha}\right) d\alpha. \end{aligned}$$

Therefore, $\widehat{v} \in h(\{x \in \mathbf{R} \mid a < x < b\}$, and we can conclude that $\widehat{v} \in h(W)$, as desired. $\square$

We remark after this lemma that it is remarkable that $\widehat{v}$ is any smoother than continuous (recall that harmonic functions are C^∞, by **Lemma 2.3.2**. Consider a similar construction in $\mathbf{R}$:

Example 2.3.16. Let $f(x) = x^2$ on $[0,1]$. Then, using the same procedure as in **Lemma 2.3.15**, we produce

$$\widehat{f}(x) = \begin{cases} -x^2 & \text{if } x < 0; \\ x^2 & \text{if } x \geq 0. \end{cases}$$

And this function is clearly only C^1 at $x = 0$.

Theorem 2.3.17 (**The Schwarz Reflection Principle**). *Let $V \subseteq \mathbf{C}$ be a connected, open set such that $V \cap \mathbf{R} = \{x \in \mathbf{R} \mid a < x < b\}$. Set $U = \{z \in V \mid \operatorname{Im} z > 0\}$. Suppose that $F \in H(U)$, and that*

$$\lim_{\substack{z \to x \\ z \in U}} \operatorname{Im} F(z) = 0$$

for each $x \in V \cap \mathbf{R}$. Define $\widehat{U} = \{z \in \mathbf{C} \mid \overline{z} \in U\}$. Then, there exists $G \in H\left(U \cup \widehat{U} \cup \{x \in \mathbf{R} \mid a < x < b\}\right)$ satisfying $G|_U \equiv F$.

In particular,

$$\varphi(x) = \lim_{\substack{z \to x \\ z \in U}} \operatorname{Re} F(z)$$

exists for each $x = x + i0 \in V \cap \mathbf{R}$, and

$$G(z) = \begin{cases} F(z) & \text{if } z \in U; \\ \varphi(x) + i0 & \text{if } z \in \{x \in \mathbf{R} \mid a < x < b\}; \\ \overline{F(\overline{z})} & \text{if } z \in \widehat{U}. \end{cases}$$

Before we prove the **Schwarz Reflection Principle 2.3.17**, notice a few striking facts:

1. If

$$F(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j,$$

then

$$F(\overline{z}) = \sum_{j=0}^{\infty} a_j (\overline{z} - z_0)^j,$$

and

$$\overline{F(\overline{z})} = \sum_{j=0}^{\infty} \overline{a_j} \overline{(\overline{z} - z_0)}^j = \sum_{j=0}^{\infty} \overline{a_j} (z - \overline{z_0})^j,$$

which is indeed holomorphic near $\overline{z_0}$.

2. We have not in the hypotheses even assumed that

$$\varphi(x) = \lim_{\substack{z \to x \\ z \in U}} \operatorname{Re} F(z)$$

exists, so it is not clear that G is continuous, or even that it is well-defined.

3. In order for G to exist, we are forced to have that $G(z) = \overline{F(\overline{z})}$ on $\widehat{U}$, and $\overline{F(\overline{z})}$ is holomorphic on $\widehat{U}$. Thus, the only necessary part of this proof will be to check the behavior of G on $\{x \in \mathbf{R} \mid a < x < b\}$.

Let us now prove the **Schwarz Reflection Principle 2.3.17**.

Proof. Let $x \in \{x \in \mathbf{R} \mid a < x < b\}$. Choose $\varepsilon > 0$ so that $D(x,\varepsilon) \subseteq U \cup \widehat{U} \cup \{x \in \mathbf{R} \mid a < x < b\}$. Set $v(z) = \operatorname{Im} F(z)$ for $z \in D(x,\varepsilon) \cap U$. By hypothesis, for all $t \in \{x \in \mathbf{R} \mid a < x < b\}$,

$$\lim_{\substack{z \to t \\ z \in D(x,\varepsilon)\cap U}} v(z) = 0,$$

and $v \in h(D(x,\varepsilon) \cap U)$.

Thus, by **Lemma 2.3.15**, there exists $\widehat{v} \in h(D(x,\varepsilon))$ so that $\widehat{v} = v$ on $D(x,\varepsilon) \cap U$. Choose $\widehat{u} \in h(D(x,\varepsilon))$ so that $\widehat{u} + i\widehat{v} \in H(D(x,\varepsilon))$. (Recall that this is possible via **Lemma 2.3.3**; disks allow us to find harmonic conjugates.)

On $D(x,\varepsilon) \cap U$, $\operatorname{Im}(F - (\widehat{u} + i\widehat{v})) = \operatorname{Im} F - \widehat{v} = \operatorname{Im} F - v = 0$. Only a constant holomorphic function can have a zero imaginary part on a connected open set.

> (This was a homowork problem, and thus hasn't appeared in these notes. But see quickly that if $F = u + iv \in H(U)$, then F satisfies the **Cauchy-Riemann equations 1.5.4**, so $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. If F is real valued, then $v \equiv 0$, so $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0$. As U is connected, $F \equiv C$.)

We can conclude that $F = \widehat{u} + i\widehat{v} + C$ for some $C \in \mathbf{R}$ fixed.

Set $G_0 = \widehat{u} + i\widehat{v} + C$ so that $G_0 \in H(D(x,\varepsilon))$ and $G_0 \equiv F$ on $D(x,\varepsilon) \cap U$.

We have now seen that F has a holomorphic extension to G_0 on $D(x,\varepsilon)$. The function $\lambda(z) = \overline{G_0(\overline{z})}$ is also holomorphic on $D(x,\varepsilon)$. Since G_0 is real valued on $D(x,\varepsilon) \cap \{x \in \mathbf{R} \mid a < x < b\}$, so is λ, and in fact, $\lambda(z) = G_0(z)$ there. Hence, $G_0(z) = \overline{G_0(\overline{z})}$ on all of $D(x,\varepsilon)$.

Therefore, by the arbitrary nature of x, the function G defined by the statement of the theorem is holomorphic on $U \cup \widehat{U} \cup \{x \in \mathbf{R} \mid a < x < b\}$. $\square$

Corollary 2.3.18. *Let $F \in C(\overline{D(0,1)}) \cap H(D(0,1))$. Suppose there exists an open arc $I \subseteq \partial D(0,1)$ such that $F|_I \equiv 0$. Then $F \equiv 0$.*

Proof. First, if $I = \partial D(0,1)$, then the result follows by the **Maximum Modulus Theorem 1.17.16**. Thus, we may assume without loss of generality that $I \subsetneq \partial D(0,1)$, and in particular, that $-1 \notin I$ (a rotation will let us do so).

Let $\varphi : \overline{D(0,1)} \to \overline{U}$, where $U = \{z \in \mathbf{C} \mid \operatorname{Im} z > 0\}$, and consider $\overline{U}$ to be the closure in the Riemann sphere $\widehat{\mathbf{C}}$; i.e., $\overline{U} = U \cup \{z = x + iy \mid y = 0\} \cup \{\infty\}$. Define φ to be the Cayley transform, $\varphi(z) = i \cdot \frac{1-z}{1+z}$ (recall, if you like, the **inverse Cayley transform 2.1.14**).

Then $G = F \circ \varphi^{-1} \in H(U) \cap C(\overline{U})$, and $G|_{\varphi(I)} \equiv 0$. Now, the fact that $\varphi(I)$ is an interval follows from the connectedness of I and the continuity of φ on $\partial D(0,1)$.

So let $V \subseteq U$ be an open half-disk with $\partial V \cap \mathbf{R} = \varphi(I)$. We may use the **Schwarz Reflection Principle 2.3.17** to Schwarz-reflect G to $\widehat{G} \in H\left(V \cup \widehat{V} \cup \varphi(I)\right)$. But since $\widehat{G}$ is holomorphic and is identically zero on $\varphi(I)$, $\widehat{G}$ has an accumulation point of zeros inside its domain. Thus, by **Theorem 1.13.2**, $\widehat{G} \equiv 0$, so G as well, and $F \equiv 0$, as desired. $\square$

Now we turn to results by Harnack regarding harmonic functions. We know that harmonic functions are the real part of holomorphic functions, and, as such, many properties for holomorphic functions have analogues for harmonic functions. **Harnack's Principle 2.3.21** is akin to **Montel's Theorem 2.2.6**. We have first an interesting inequality.

Lemma 2.3.19 (Harnack's Inequality). *Let $R > 0$ and u be harmonic and positive (nonnegative suffices, by the **Minimum Prinicple 2.3.5**) on a neighborhood of $D(0,R)$. Then, for any $z \in D(0,R)$, we have*

$$\frac{R - |z|}{R + |z|} u(0) \le u(z) \le \frac{R + |z|}{R - |z|} u(0).$$

Proof. By the **Poisson Integral Formula 2.3.9**, if $z \in D(0,R)$, then

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u\left(Re^{i\psi}\right) \frac{R^2 - |z|^2}{\left|Re^{i\psi} - z\right|^2} d\psi.$$

Observe that

$$\frac{R^2-|z|^2}{\left|Re^{i\psi}-z\right|^2} \le \frac{R^2-|z|^2}{\left(R-|z|\right)^2} = \frac{\left(R-|z|\right)\left(R+|z|\right)}{\left(R-|z|\right)^2} = \frac{R+|z|}{R-|z|}.$$

From the above, one can see that

$$u(z) \le \frac{1}{2\pi}\int_0^{2\pi} u\left(Re^{i\psi}\right)\frac{R+|z|}{R-|z|}d\psi = u(0)\frac{R+|z|}{R-|z|}.$$

For the other half of the result, we replace the estimate with

$$\frac{R^2-|z|^2}{\left|Re^{i\psi}-z\right|^2} \ge \frac{R^2-|z|^2}{\left(R+|z|\right)^2} = \frac{\left(R-|z|\right)\left(R+|z|\right)}{\left(R+|z|\right)^2} = \frac{R-|z|}{R+|z|},$$

and use the same argument. □

Corollary 2.3.20. *Let u be a nonnegative harmonic function on a neighborhood of $\overline{D(P,R)}$. Then*

$$\frac{R-|z-P|}{R+|z-P|}u(P) \le u(z) \le \frac{R+|z-P|}{R-|z-P|}u(P)$$

for any $z \in D(P,R)$.

Proof. This is just a transformation applied to **Harnack's Inequality 2.3.19**. □

Theorem 2.3.21 (**Harnack's Principle**). *Let $u_1 \le u_2 \le u_3 \le \cdots$ be a sequence of harmonic functions in $U \subseteq \mathbf{C}$, where U is a connected, open set. Then, either $u_j \to \infty$ uniformly on compact sets, or there exists a harmonic function $u \in h(U)$ for which $u_j \to u$ normally.*

Note that this theorem is rather powerful; if $u_1 \le u_2 \le \cdots$, and $\{u_j(z)\}$ is bounded for a *single* z, then $u_j \to u$ normally.

Proof. If $P \in U$ and $u_j(P) \to \infty$, then for some j_0, $u_{j_0}(P) > 0$. Thus, by continuity, there exists some $r > 0$ for which $\overline{D(P,r)} \subseteq U$ and $u_{j_0} > 0$ on $\overline{D(P,r)}$. By **Harnack's Inequality 2.3.19**, for $z \in D\left(P, \frac{r}{2}\right)$,

$$u_j(z) \ge \frac{r-\frac{r}{2}}{r+\frac{r}{2}}u_j(P) = \frac{1}{3}u_j(P) \to \infty$$

for $j \ge j_0$. Thus, $u_j \to \infty$ uniformly on $\overline{D\left(P,\frac{r}{2}\right)}$.

On the other hand, if $Q \in U$ and $u_j(Q) \to \ell < \infty$ as $j \to \infty$, then choose s so that $\overline{D(Q,s)} \subseteq U$. Again by **Harnack's Inequality 2.3.19**, if $z \in \overline{D\left(Q,\frac{s}{2}\right)}$, then if $j > k$,

$$u_j(z) - u_k(z) \le \frac{s+\frac{s}{2}}{s-\frac{s}{2}}\left(u_j(Q)-u_k(Q)\right) = 3\left(u_j(Q)-u_k(Q)\right) \to 0$$

as $j,k \to \infty$. Thus, (u_j) converges uniformly on $\overline{D\left(Q,\frac{s}{2}\right)}$ to some harmonic u.

We have therefore established that both $\{z \in U \mid u_j(z) \to \infty\}$ and $\{z \in U \mid u_j(z) \to u(z)\}$ are open, obviously disjoint, and whose union is U. Since U is connected, at least one must be empty. This verifies the alternatives in the statement of the theorem.

It remains to show that convergence is uniform on compact subsets of U. However, this is easy; given $K \subseteq U$ compact, K may be covered by finitely many balls of radius $\frac{r}{2}$ or $\frac{s}{2}$. Hence, the convergence is uniform. □

2.4 The Dirichlet Problem and Subharmonic Functions

Definitions: convex, subharmonic, barrier

Main Idea: We have seen that the Dirichlet problem gives us an open set U and a continuous function f on ∂U, and asks us to find a harmonic function u in U that agrees with f on the boundary. In this section, we introduce subharmonic functions (perhaps best thought of as satisfying the submean value property, as their actual definition is rather unwieldly and only serves to justify the similarity to convexity). We prove some results about subharmonic functions, and then introduce the concept of a barrier function for U at a point $P \in \partial U$. The existence of a barrier at a point allows us to solve the Dirichlet problem on U; such a theorem is due to Perròn. Finally, we briefly mention a result first mentioned in discussing the Riemann Mapping Theorem; it tells us nothing when the sets we compare are not simply connected, so we see the conformal equivalence conditions for two annuli.

We deviate slightly from harmonic functions to explore subharmonic functions, and see how their application helps solve the Dirichlet problem on open sets other than $D(0,1)$ (as in **Theorem 2.3.10**).

Given $U \subseteq \mathbf{C}$ and $f \in C(\partial U)$, the Dirichlet problem is the question of finding $u \in C(\overline{U})$ such that $\Delta u = 0$ in U and $u \equiv f$ on ∂U. We have already solved the Dirichlet problem on the unit disk, $D(0,1)$, in **Theorem 2.3.10**. If U is bounded, we can easily show that if a solution exists, then it must be unique (simply take their difference and apply the **Maximum Theorem 2.3.6**).

If, however, U is not nice (with later explanation of what we mean by "nice;" see **barriers 2.4.15** and the **Perròn method 2.4.21**), then there may not be any solution to the Dirichlet problem. Consider the following example.

Example 2.4.1. Let $U = D(0,1) \setminus \{0\}$. Then $\partial U = \{z \mid |z| = 1\} \cup \{0\}$. Now set

$$f(z) = \begin{cases} 1 & \text{if } |z| = 1; \\ 0 & \text{if } z = 0. \end{cases}$$

Then, $f \in C(\partial U)$. So we could try to solve the Dirichlet problem here.

We claim that if $u \in C(\overline{U})$ and u solves the Dirichlet problem, then u must be radial. To see this, one can perform the computation that verifies that if $u(z)$ is a solution, then so is $u\left(ze^{i\theta}\right)$ for any fixed θ. (*One* can; I will not. Take a Laplacian.)

By uniqueness, $u(z) = u\left(ze^{i\theta}\right)$.

We can write Δ in polar coordinates; one can check that

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Since u is radial, $\frac{\partial u}{\partial \theta} = 0$. Thus,

$$0 = \Delta u = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) u = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u}{\partial r}\right].$$

Thus, $r\frac{\partial u}{\partial r} \equiv C$, which implies that $\frac{\partial u}{\partial r} \equiv \frac{C}{r}$. Thus, $u = C \log r + D$, but no values of C and D can satisfy the boundary conditions. Thus the Dirichlet problem cannot be solved on the punctured disk with the given boundary function.

So not every Dirichlet problem can be solved. It must be the case, therefore, that some conditions on U (or, less importantly, on f) are necessary. We'll see that if ∂U consists of smooth enough curves, then the Dirichlet problem has a unique solution.

To reach this goal, we turn to a discussion of subharmonicity. This is a complex analysis version of convexity.

Example 2.4.2. We begin our discussion with real valued functions on $\mathbf{R}$. On $\mathbf{R}$, we know that $\Delta = \frac{d^2}{dx^2}$. If $\Delta f \equiv 0$, then $f(x) = cx + d$. Now set

$$\mathcal{S} = \left\{ f \in C(\mathbf{R}) \mid \text{ if } [a,b] \subseteq \mathbf{R}, h \in h([a,b]), f(a) \le h(a), \text{ and } f(b) \le h(b), \text{ then } f(x) \le h(x) \text{ for all } x \in [a,b] \right\}.$$

We naturally ask: what functions are in $\mathcal{S}$? The answer is that $\mathcal{S}$ is the set of convex (i.e., concave up, in calculus 1) functions.

Definition 2.4.3. A function $f : [a, b] \to \mathbf{R}$ is **convex** if whenever $[c, d] \subseteq [a, b]$ and $0 \leq \lambda \leq 1$, then

$$f\big((1-\lambda)c + \lambda d\big) \leq (1-\lambda)f(c) + \lambda f(d).$$

Note that, as per the definition, we can discuss convexity in the absence of differentiability. So this is of course slightly stronger than calculus 1's concavity.

As we've stated earlier, subharmonicity is a complex analysis version of convexity. See so in the following definition.

Definition 2.4.4. Let $U \subseteq \mathbf{C}$ be open, and let $f \in C(U, \mathbf{R})$. Suppose that for each $\overline{D(P, r)} \subseteq U$ and every harmonic function h defined on a neighborhood of $\overline{D(P, r)}$ that satisfies $f \leq h$ on $\partial D(P, r)$, it then holds that $f \leq h$ on $D(P, r)$ as well. Then f is called **subharmonic** on U.

We write $sh(U) = \{f : U \to \mathbf{R} \mid f \text{ is subharmonic on } U\}$.

Lemma 2.4.5. *Harmonic functions are subharmonic.*

Proof. To see this, let $U \subseteq \mathbf{C}$ be open, and let $u \in h(U)$. Let $\overline{D(P, r)} \subseteq U$, and let h be harmonic on a neighborhood of $\overline{D(P, r)}$ so that $u \leq h$ on $\partial D(P, r)$. Then, $u - h \leq 0$ on $\partial D(P, r)$. By the **Maximum Principle 2.3.4**, $u - h \leq 0$ on $D(P, r)$. Thus, $u \leq h$ on $D(P, r)$, so $u \in sh(U)$. ☐

Proposition 2.4.6. *Let $U \subseteq \mathbf{C}$ be open, and let $f \in sh(U) \cap C^2(U)$. Then $\Delta f \geq 0$.*

The only way I know to prove this uses a lot of machinery from partial differential equations. It certainly can be proven, but not without a large tangent explaining the tools we'd need. Thus, it remains a proposition in these notes.

Lemma 2.4.7. *Let $U \subseteq \mathbf{C}$ be open, and let $f \in C(U, \mathbf{R})$. Suppose that for $\overline{D(P, r)} \subseteq U$,*

$$f(P) \leq \frac{1}{2\pi} \int_0^{2\pi} f\left(P + re^{i\theta}\right) d\theta.$$

We call this the submean value property[9]*. Then $f \in sh(U)$. Conversely, if $f \in sh(U)$ and $\overline{D(P, r)} \subseteq U$, then the submean value property holds.*

Proof. First, suppose that the submean value property holds for every P and r as in the hypothesis, but that $f \notin sh(U)$. Then, there exists a disk $\overline{D(Q, s)} \subseteq U$ and a harmonic function h defined on a neighborhood of $\overline{D(Q, s)}$ such that $f \leq h$ on $\partial D(Q, s)$, but $f(z_0) > h(z_0)$ for some $z_0 \in D(Q, s)$.

Let $g = f - h$ on $\overline{D(Q, s)}$. The function g satisfies $g \leq 0$ on $\partial D(Q, s)$ and $g(z_0) > 0$. Set $M = \max\limits_{\overline{D(Q,s)}} g$, and $K = \{z \in \overline{D(Q, s)} \mid g(z) = M\}$. Now, K is a compact subset of $D(Q, s)$, so therefore $K \neq D(Q, s)$. Let $w \in \partial K$. Then there exists $\eta > 0$ so that $D(w, \eta) \subseteq D(Q, s)$, and there is a point of $\partial D(w, \eta)$ at which $g < M$. By continuity, there is an open arc $J \subseteq \partial D(w, \eta)$ on which $g < M$. Therefore, for this η,

$$\frac{1}{2\pi} \int_0^{2\pi} g\left(w + \eta e^{i\theta}\right) d\theta < M = g(w).$$

But see that

$$\begin{aligned}\frac{1}{2\pi} \int_0^{2\pi} g\left(w + \eta e^{i\theta}\right) d\theta &= \frac{1}{2\pi} \int_0^{2\pi} f\left(w + \eta e^{i\theta}\right) d\theta - \frac{1}{2\pi} \int_0^{2\pi} h\left(w + \eta e^{i\theta}\right) d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f\left(w + \eta e^{i\theta}\right) d\theta - h(w),\end{aligned}$$

[9]If the inequality were to go in the other direction, we would call that the supermean value property.

and that

$$g(w) = f(w) - h(w),$$

so the inequality can be realized as

$$\frac{1}{2\pi}\int_0^{2\pi} f\left(w+\eta e^{i\theta}\right)d\theta - h(w) < f(w) - h(w), \text{ i.e.,}$$
$$\frac{1}{2\pi}\int_0^{2\pi} f\left(w+\eta e^{i\theta}\right)d\theta < f(w).$$

This contradicts the fact that the submean value property holds for every disk in U, so $f \in sh(U)$.

For the converse, let $f \in sh(U)$, and let $\overline{D(Q,s)} \subseteq U$. Let $P : \overline{D(Q,s)} \times \partial D(Q,s) \to \mathbf{R}$ be the Poisson kernel for $D(Q,s)$ (recall the **Poisson Integral Formula 2.3.9**). Recall that the Poisson kernel is positive.

Let $\varepsilon > 0$. Then define

$$h(z) = \int_0^{2\pi} P\left(z, Q+se^{i\theta}\right)\cdot\left(f\left(Q+se^{i\theta}\right)+\varepsilon\right)d\theta.$$

Then $h \in h(D(Q,s)) \cap C(\overline{D(Q,s)})$. Also, $h(\zeta) = f(\zeta)+\varepsilon > f(\zeta)$ for $\zeta \in \partial D(Q,s)$. It therefore follows that $h(\zeta) > f(\zeta)$ for $\zeta \in \partial D(Q,s-\delta)$ if δ is small enough, since f and h are continuous. But h is harmonic near $\overline{D(Q,s-\delta)}$, so by the subharmonicity of f, $f \le h$ on $D(Q,s-\delta)$. In particular, $f(Q) \le h(Q)$, and

$$h(Q) = \frac{1}{2\pi}\int_0^{2\pi}\left(f\left(Q+se^{i\theta}\right)+\varepsilon\right)d\theta.$$

Send $\varepsilon \to 0$ to complete the proof. □

Lemma 2.4.8. *Let $F \in H(U)$. Then $|F| \in sh(U)$.*

Proof. To see this, let $\overline{D(w,s)} \subseteq U$. By the **Mean Value Property 2.3.7** (since holomorphic functions are harmonic), we get

$$|F(Q)| = \left|\frac{1}{2\pi}\int_0^{2\pi} F\left(Q+se^{i\theta}\right)d\theta\right| \le \frac{1}{2\pi}\int_0^{2\pi}\left|F\left(Q+se^{i\theta}\right)\right|d\theta.$$

By **Lemma 2.4.7**, $|F| \in sh(U)$. □

Note, however, that typically $|F| \notin h(U)$.

Example 2.4.9. Consider

$$\frac{\partial^2}{\partial z\partial\overline{z}}|z|^k = \frac{\partial^2}{\partial z\partial\overline{z}}\left[z^{\frac{k}{2}}\overline{z}^{\frac{k}{2}}\right] = \frac{\partial}{\partial z}\left[\frac{k}{2}z^{\frac{k}{2}}\overline{z}^{\frac{k}{2}-1}\right] = \left(\frac{k}{2}\right)^2|z|^{k-2},$$

which is typically not zero.

Lemma 2.4.10. *Let $f \in sh(U)$, and let $\varphi : \mathbf{R} \to \mathbf{R}$ be nondecreasing and convex. Then $\varphi \circ f \in sh(U)$.*

Proof. Let $\overline{D(P,r)} \subseteq U$. Then

$$\varphi(f(P)) \le \varphi\left(\frac{1}{2\pi}\int_0^{2\pi} f\left(P+re^{i\theta}\right)d\theta\right).$$

And by Jensen's inequality[10], we have

$$\varphi\left(\frac{1}{2\pi}\int_0^{2\pi} f\left(P+re^{i\theta}\right)d\theta\right) \le \frac{1}{2\pi}\int_0^{2\pi}\varphi\left(f\left(P+re^{i\theta}\right)\right)d\theta.$$

Hence $\varphi \circ f \in sh(U)$. □

[10]For the proof of Jensen's inequality, see Real Analysis.

Example 2.4.11. If $f \in sh(U)$, then $e^f \in sh(U)$.

Example 2.4.12. If $f \in sh(U)$ and $f \geq 0$, then $f^2 \in sh(U)$.

Note that in **Lemma 2.4.10**, if $f \in h(U)$, then only convex is needed:

$$\varphi(f(P)) = \varphi\left(\frac{1}{2\pi}\int_0^{2\pi} f\left(P + re^{i\theta}\right) d\theta\right) \leq \frac{1}{2\pi}\int_0^{2\pi} \varphi\left(f\left(P + re^{i\theta}\right)\right) d\theta,$$

by Jensen. So $\varphi \circ f \in sh(U)$.

Lemma 2.4.13 (**Maximum Principle for Subharmonic Functions**). *If $f \in sh(U, \mathbf{R})$ and if there exists $P \in U$ such that $f(P) \geq f(z)$ for all $z \in U$, then f is constant.*

Proof. The proof is identical to the proof for harmonic functions (**Maximum Principle 2.3.4**). □

Note that subharmonic functions don't have a minimum principle; superharmonic functions, however, do.

Lemma 2.4.14. *Let $U \subseteq \mathbf{C}$ and let $f_1, f_2 \in sh(U)$. Then*

1. *$f_1 + f_2 \in sh(U)$,*
2. *if $\alpha > 0$, $\alpha f \in sh(U)$, and*
3. *$g(z) = \max\{f_1(z), f_2(z)\} \in sh(U)$.*

Proof. 1. and 2. follow immediately from the submean value property (**Lemma 2.4.7**).

To see 3., see that

$$\begin{aligned} g(P) = \max\{f_1(P), f_2(P)\} &\leq \max\left\{\frac{1}{2\pi}\int_0^{2\pi} f_1\left(P + re^{i\theta}\right) d\theta, \frac{1}{2\pi}\int_0^{2\pi} f_2\left(P + re^{i\theta}\right) d\theta\right\} \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} \max\left\{f_1\left(P + re^{i\theta}\right), f_2\left(P + re^{i\theta}\right)\right\} d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} g\left(P + re^{i\theta}\right) d\theta. \end{aligned}$$

Thus by **Lemma 2.4.7**, $g \in sh(U)$. □

The following concept is of fundamental importance in partial differential equations. We will see that it gives us a sufficient condition for solving the Dirichlet problem.

Definition 2.4.15. Let $U \subseteq \mathbf{C}$ be open, and let $P \in \partial U$. A function $b : \overline{U} \to \mathbf{R}$ is called a **barrier** for U at P if:

1. b is continuous on $\overline{U}$,
2. b is subharmonic on U,
3. $b \leq 0$, and
4. $\{z \in \partial U \mid b(z) = 0\} = \{P\}$.

Note that by the **Maximum Principle for Subharmonic Functions 2.4.13**, $b(z) = 0$ on $\overline{U}$ if and only if $z = P$, and therefore, $b(z) < 0$ whenever $z \in \overline{U} \setminus \{P\}$.

Example 2.4.16. If $U = D(0, 1)$ and $P = 1 + i0$, then the function $b(z) = \operatorname{Re} z - 1$ is a barrier for U at 1.

Similarly, we can construct the following:

Example 2.4.17. Let $U \subseteq \mathbf{C}$ be a bounded, open set. Let $P \in \overline{U}$ be a point in $\overline{U}$ that is furthest from the origin. Let $r = |P| = |P - 0|$. Then $\overline{U} \subseteq \overline{D(0,r)}$, and note that $P \in \partial D(0,r)$. Set $\theta_0 = \arg P$ for $\theta_0 \in [0, 2\pi]$. Then the function $b(z) = \operatorname{Re}\left(e^{-i\theta_0}z\right) - r$ is a barrier for U at P.

Example 2.4.18. Let $U \subseteq \mathbf{C}$ be an open set. Let $P \in \partial U$. Suppose further that there exists a closed line segment I, the line from Q to P, such that $Q \in \mathbf{C} \setminus U$ and $I \cap \overline{U} = \{P\}$. Then, the function $z \mapsto \frac{z-P}{z-Q}$ maps $\mathbf{C} \setminus I$ to $\mathbf{C} \setminus J$, where J is a closed, infinite ray from the origin.

Therefore, $\psi(z) = \sqrt{\frac{z-P}{z-Q}}$ has a well-defined branch on $U \subseteq \mathbf{C} \setminus I$, and extends to a continuous, one-to-one map of $\overline{U}$ into $\mathbf{C}$. Note that $\psi(\overline{U})$ is contained in a closed half plane.

Finally, a suitable rotation ρ_θ maps $\psi(\overline{U})$ to $\{z \mid \operatorname{Im} z > 0\}$, and composing with the Cayley transform φ maps $\rho_\theta \circ \psi(\overline{U})$ to $D(0,1)$, with $\varphi \circ \rho_\theta \circ \psi(P) = 1$, since $\psi(P) = 0$.

Consequently, $\varphi \circ \rho_\theta \circ \psi : \overline{U} \to \overline{D(0,1)}$, and $\varphi \circ \rho_\theta \circ \psi(P) = 1$; thus this example reduces to the case of the unit disk, for which we have already constructed a barrier (**Example 2.4.16**). Thus the barrier for U at P is $b(z) = \operatorname{Re}\left((\varphi \circ \rho_\theta \circ \psi)(z)\right) - 1$.

Example 2.4.18 is of paramount importance, because it follows that if U is bounded and bounded by a smooth curve (say, of class C^1), then every point of the boundary of U has a barrier.

Lemma 2.4.19. *If $b : \overline{U} \cap D(P,r) \to \mathbf{R}$ is a barrier for $U \cap D(P,r)$ at P, then we can construct a barrier on the entire domain; i.e., there exists a barrier function $\widetilde{b} : \overline{U} \to \mathbf{R}$ for U at P.*

Proof. Let $U \subseteq \mathbf{C}$ be open, and let $P \in \partial U$. Suppose that for some $r > 0$, there is a barrier at P for $U \cap D(P,r)$.

If b is such a barrier, then define $\widetilde{b} : \overline{U} \to \mathbf{R}$ by

$$\widetilde{b}(z) = \begin{cases} -\varepsilon & \text{if } z \in \overline{U} \setminus D(P,r); \\ \max\{-\varepsilon, b(z)\} & \text{if } z \in \overline{U} \cap D(P,r). \end{cases}$$

Then, $\widetilde{b}$ is a barrier if $\varepsilon > 0$ is suitably small.

Therefore, the existence of a barrier is a local property. ☐

Now, we have claimed above that barriers give us a way to solve the Dirichlet problem. We will do so soon (the **Perròn method 2.4.21**), but to drive this idea home first, we will show the nonexistence of a barrier function for a set for which we already know we cannot solve the Dirichlet problem.

Example 2.4.20. Let $U = D(0,1) \setminus \{0\}$. We will show that no barrier exists at 0.

Assume that we have a barrier function b. Then, by definition, $b \in C(\overline{U}) \cap sh(U)$, $b \leq 0$ on $\overline{U}$, and $\{z \in \overline{U} \mid b(z) = 0\} = \{0\}$.

We claim that the function

$$\widehat{b}(z) = \frac{1}{2\pi} \int_0^{2\pi} b\left(e^{i\theta}z\right) d\theta$$

is also a barrier for U at 0.

To see that $\widehat{b} \in C(\overline{U})$, that $\widehat{b} \leq 0$ on $\overline{U}$, and that $\{z \in \overline{U} \mid \widehat{b}(z) = 0\} = \{0\}$ is straightforward.

We show that $\widehat{b} \in sh(U)$ by showing that $\widehat{b}$ satisfies the submean value property (**Lemma 2.4.7**). Pick $z \in U$ and $r > 0$ such that $\overline{D(z,r)} \subseteq U$.Note then that $\overline{D\left(ze^{i\theta}, r\right)} \subseteq U$ as well. Then

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \widehat{b}\left(z + re^{i\psi}\right) d\psi &= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2\pi} \int_0^{2\pi} b\left(\left(z + re^{i\psi}\right) e^{i\theta}\right) d\theta d\psi \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2\pi} \int_0^{2\pi} b\left(ze^{i\theta} + re^{i(\psi+\theta)}\right) d\psi d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2\pi} \int_0^{2\pi} b\left(ze^{i\theta} + re^{i\psi'}\right) d\psi' d\theta \\ &\geq \frac{1}{2\pi} \int_0^{2\pi} b\left(ze^{i\theta}\right) d\theta \\ &= \widehat{b}(z). \end{aligned}$$

Thus, $\widehat{b} \in sh(U)$, and $\widehat{b}$ is a barrier for U at 0.

The function $\widehat{b}$ is radial; i.e., $\widehat{b}(z) = \widehat{b}\left(ze^{i\psi}\right)$ for any $|\psi| = 1$. Thus, $\widehat{b}\left(e^{i\psi}\right)$ is a negative constant. Scaling b does not change the fact that b is a barrier (i.e., $b \mapsto \alpha b$, $\alpha > 0$), so the existence of b means that there exists another barrier B on $\overline{U}$ so that $B|_{\partial D(0,1)} \equiv -1$, and B is radial.

We will show that this is impossible.

Set $H_r(z) = \frac{B(r)+1}{\log r} \log|z| - 1$ for $r \in (0,1)$. Then H_r is harmonic for all fixed r. (To check, it helps to write $\log|z| = \frac{1}{2}\log|z|^2 = \frac{1}{2}\log(z\overline{z})$. Then compute Δ.) So $H_r \in h(U)$, $H_r\left(e^{i\theta}\right) = -1$, and $H_r\left(re^{i\theta}\right) = B(r)$, so H_r agrees with B on two circles: $\{z \mid |z| = 1\}$ and $\{z \mid |z| = r\}$.

B is subharmonic, so $B - H_r$ is as well, by **Lemma 2.4.5** and **Lemma 2.4.14**. Since $B - H_r \equiv 0$ on $\{z \mid |z| = 1 \text{ or } |z| = r\}$, it follows that for $z \in \{w \mid r \le |w| \le 1\}$, $B(z) \le H_r(z)$. Thus, for each fixed $z \in U$,

$$B(z) \le \lim_{r \to 0^+} H_r(z) = \lim_{r \to 0^+} \frac{B(r)+1}{\log r} \log|z| - 1 = -1,$$

since $B(r)$ is bounded on $\overline{U}$ and $\log r \to -\infty$ as $r \to 0^+$. This contradicts the fact that

$$\lim_{z \to 0} B(z) = 0.$$

Thus, $U = D(0,1) \setminus \{0\}$ has no barrier at 0.

We now solve the Dirichlet problem using subharmonic functions and barrier, via the **Perròn method 2.4.21**:

Theorem 2.4.21 (**Perròn**). *Let $U \subseteq \mathbf{C}$ be a bounded, connected, open set that has a barrier b_P for each $P \in \partial U$. Then the Dirichlet problem can be solved on U. This means that given $f \in C(\partial U)$, there exists a unique function $u \in C(\overline{U}) \cap h(U)$ such that $u|_{\partial U} \equiv f$.*

Proof. Without loss of generality, we may take f to be real valued. Also, the uniqueness statement (as noted in the start of this section) is an immediate consequence of the **Maximum Theorem 2.3.6**.

We turn to the (harder) question of existence. Set

$$\mathcal{S} = \left\{ \psi \in sh(U) \mid \limsup_{\substack{z \to P \\ z \in U}} \psi(z) \le f(P) \text{ for all } P \in \partial U \right\}.$$

As ∂U is compact and $f \in C(\partial U)$, there exists $m \in \mathbf{R}$ so that $f \ge m$ on ∂U. Thus $\psi(z) \equiv m \in \mathcal{S}$, so $\mathcal{S}$ is not empty.

For $z \in U$, set

$$u(z) = \sup_{\psi \in \mathcal{S}} \psi(z).$$

We claim that u solves the Dirichlet problem. We will show as much in three parts. In the first, we show that u is bounded above. In the second, we show that u is harmonic on U. In the third, we show that if $w \in \partial U$, then $\lim_{\substack{z \to w \\ z \in U}} u(z) = f(w)$.

Part I: We show that u is bounded above.

Set $M = \max_{\zeta \in \partial U} f(\zeta)$. Let $\psi \in \mathcal{S}$, $\varepsilon > 0$, and $E_\varepsilon = \{z \in U \mid \psi(z) \ge M + \varepsilon\}$. We will show that $E_\varepsilon = \emptyset$.

We claim that if $E_\varepsilon \ne \emptyset$, then E_ε is closed. We'll show its complement is open. There are three possibilities to consider:

1. If $z \in \mathbf{C} \setminus \overline{U}$, then $\mathbf{C} \setminus \overline{U}$ is open, as $\overline{U}$ is closed, so there exists a neighborhood containing $z \in \mathbf{C} \setminus \overline{U} \subseteq \mathbf{C} \setminus E_\varepsilon$.

2. If $z \in \partial U$, then we use the fact that $f(z) \leq M$. Since $\psi \in \mathcal{S}$, there exists a neighborhood V of z in which $\psi(w) \leq M + \varepsilon$ for all $w \in V \cap U$. Thus, $V \cap E_\varepsilon = \emptyset$, so $\mathbf{C} \setminus E_\varepsilon$ is again open.

3. Finally, if $z \in U \setminus E_\varepsilon$, then $\psi(z) < M + \varepsilon$. By the continuity of ψ on U, there exists a neighborhood V of z on which $\psi(w) \leq M + \varepsilon$ for all $w \in V$.

In all three cases, $\mathbf{C} \setminus E_\varepsilon$ is open, and thus E_ε is closed.

Since $E_\varepsilon \subseteq U$, E_ε is bounded. So E_ε is both closed and bounded; by Heine-Borel, E_ε is a compact set. As $\psi \in C(U)$, $\psi \in C(E_\varepsilon)$, and therefore ψ takes a maximum on E_ε; call it $P \in E_\varepsilon$. Since $P \in E_\varepsilon$, $\psi(P) \geq M + \varepsilon$. If $z \in \mathbf{C} \setminus E_\varepsilon \cap U$, then $\psi(z) < M + \varepsilon$, so

$$\psi(P) = \max_{z \in U} \psi(z).$$

By the **Maximum Principle 2.3.4**, $\psi(z) \equiv \psi(P)$ for all $z \in U$. This is a contradiction, since we have $M + \varepsilon < \psi(P) \leq f(P) < M$. Thus, $E_\varepsilon = \emptyset$.

Since $E_\varepsilon = \emptyset$, for all $z \in U$, $\psi(z) < M + \varepsilon$ for all $\varepsilon > 0$. Thus, for each $\psi \in \mathcal{S}$, $\psi(z) \leq M$. So

$$u = \sup_{\psi \in \mathcal{S}} \psi(z) \leq M.$$

Thus, u is bounded above by the same bound as of f (note that this is a great thing, since we're hoping to show $u \in h(U)$, so it had better respect the **Maximum Theorem 2.3.6**!).

Part II: We show that $u \in h(U)$.

Let $\overline{D(P,r)} \subseteq U$, and let $p \in D(P,r)$. By the construction of u, there exists a sequence $(\psi_j) \subseteq \mathcal{S}$ with $\psi_j(p) \to u(p)$.

Let

$$\Psi_n(z) = \max\{\psi_1(z), \psi_2(z), ..., \psi_n(z)\}$$

for each $z \in U$. Since $\psi_j \in sh(U)$, $\Psi_n \in sh(U)$ by **Lemma 2.4.14**, and $\Psi_1 \leq \Psi_2 \leq \cdots \leq \Psi_n \leq \cdots$.

Let

$$\Phi_n(z) = \begin{cases} \Psi_n(z) & \text{if } z \in U \setminus \overline{D(P,r)}; \\ \text{the Poisson integral of } \Psi_n|_{\partial D(P,r)} & \text{if } z \in \overline{D(P,r)}. \end{cases}$$

This Φ_n function is called the Poisson modification of Ψ_n. We claim that Φ_n is subharmonic on U.

To see this, we show that Φ_n satisfies the small circle submean value property. This is clearly true in $U \setminus \overline{D(P,r)}$, because Ψ_n is subharmonic, and in $D(P,r)$, as Φ_n is harmonic there. So all that remains to show is on $\partial D(P,r)$.

If $z \in \partial D(P,r)$, notice that $\Psi_n - \Phi_n$ is subharmonic in $D(P,r)$, so we have by the **Maximum Principle for Subharmonic Functions 2.4.13** that $\Psi_n(w) - \Phi_n(w) \leq \max_{z \in \partial D(P,r)} \Psi_n(z) - \Phi_n(z) = 0$. Therefore, $\Phi_n(w) \geq \Psi_n(w)$ for all $w \in D(P,r)$.

If $z \in \partial D(P,r)$ and $\overline{D(z,\varepsilon)} \subseteq U$, then

$$\Phi_n(z) = \Psi_n(z) = \frac{1}{2\pi} \int_0^{2\pi} \Psi_n\left(z + \varepsilon e^{i\theta}\right) d\theta.$$

Clearly, $\Phi_n(w) = \Psi_n(w) \geq \Psi_n(w)$ for $w \in U \setminus D(P,r)$, so

$$\frac{1}{2\pi} \int_0^{2\pi} \Psi_n\left(z + \varepsilon e^{i\theta}\right) d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \Phi_n\left(z + \varepsilon e^{i\theta}\right) d\theta,$$

so Φ_n satisfies the small circle submean value property in U, so $\Phi_n \in sh(U)$, as claimed.

Thus, $\Phi_n \in \mathcal{S}$. Furthermore, $\Phi_1(z) \leq \Phi_2(z) \leq \cdots$ for all $z \in U \setminus D(P,r)$, and by the **Maximum Principle 2.3.4** on $\Phi_{n+1} - \Phi_n$ harmonic in the disk, $\Phi_1(z) \leq \Phi_2(z) \leq \cdots$ for $z \in D(P,r)$ too.

Now,

$$\lim_{n\to\infty} \Phi_n(p) = u(p),$$

as $\Phi_n \in \mathcal{S}$, so

$$\lim_{n\to\infty} \Phi_n(p) \le u(p),$$

and as $\Phi_n \ge \Psi_n \ge \psi_n$ with $\psi_n(p) \to u(p)$,

$$\lim_{n\to\infty} \Phi_n(p) \ge u(p).$$

Thus, let

$$\Phi(z) = \lim_{n\to\infty} \Phi_n(z).$$

By **Harnack's Priniciple 2.3.21**, $\Phi \in h(D(P,r))$. We next claim that $\Phi(q) = u(q)$ for all $q \in D(P,r)$.

To see so, choose some such q. Let $\rho_j \in \mathcal{S}$ such that $\rho_j(q) \to u(q)$. Let $R_j = \max\{\rho_j(z), \psi_j(z)\}$. Define $\Lambda_n(z) = \max\{R_1(z), R_2(z), ..., R_n(z)\}$. And finally, define

$$H_n(z) = \begin{cases} \Lambda_n(z) & \text{if } z \in U \setminus \overline{D(P,r)}; \\ \text{the Poisson integral of } \Lambda_n|_{\partial D(P,r)} & \text{if } z \in D(P,r). \end{cases}$$

Then, $H_n \to H \in h(D(P,r))$, $H(q) = u(q)$, and $H(p) = u(p) = \Phi(p)$. Consider $\Phi(z) - H(z)$. Now, $\Phi - H \in h(D(P,r))$, and $\Phi(p) - H(p) = 0$. Then we know that $\Phi(z) - H(z) \le 0$ in $D(P,r)$, so as $\Phi - H$ attains its max in $D(P,r)$, by the **Maximum Principle 2.3.4**, $\Phi(z) = H(z)$ for all $z \in D(P,r)$. Thus, as claimed, $\Phi(q) = H(q) = u(q)$.

Therefore, $\Phi = u$ is harmonic locally, and by the arbitrary nature of that local $D(P,r)$, u is harmonic on all of U.

Part III: We show that if $w \in \partial U$, then $\lim\limits_{\substack{z\to w \\ z\in U}} u(z) = f(w)$.

Begin by fixing $w \in \partial U$. We will use the barrier b_w. By definition, $b_w \in C(\overline{U}) \cap sh(U)$, $b_w(z) \le 0$ for all $z \in \overline{U}$, and if $z \in \overline{U} \setminus \{w\}$, then $b_w(z) < 0$.

Let $\varepsilon > 0$. Recall that ∂U is compact, so f is uniformly continuous on ∂U. Let $\delta > 0$ be such that if $\alpha, \beta \in \partial U$ and $|\alpha - \beta| < \delta$, then $|f(\alpha) - f(\beta)| < \varepsilon$.

On $\overline{U} \setminus D(w,\delta)$, $b_w < 0$, and as $\overline{U} \setminus D(w,\delta)$ is compact, $b_w \le -\mu < 0$ on $\overline{U} \setminus D(w,\delta)$. Let

$$g(z) = f(w) + \varepsilon - \frac{M - f(w)}{\mu} b_w(z),$$

where $M = \max\limits_{\partial U} |f|$. Notice that because $-\frac{M-f(w)}{\mu} < 0$, g is superharmonic, so $-g$ is subharmonic.

We now claim that $g(\zeta) > f(\zeta)$ for all $\zeta \in \partial U$.

To prove the claim, we consider two possible cases. In the first case, consider $\zeta \in D(w,\delta)$. Then $|f(\zeta) - f(w)| < \varepsilon$, so

$$f(\zeta) < f(w) + \varepsilon \le f(w) + \varepsilon - \frac{M - f(w)}{\mu} b_w(\zeta) = g(\zeta).$$

The claim holds.

If, however, $\zeta \in \partial U \setminus D(w,\delta)$, we are in the second case. As $\zeta \in \partial U \setminus D(w,\delta) \subseteq \overline{U} \setminus D(w,\delta)$, we get that $b_w(\zeta) \le -\mu$. So

$$g(\zeta) = f(w) + \varepsilon + (M - f(w)) \frac{b_w(\zeta)}{-\mu} \le f(w) + \varepsilon + M - f(w) > f(\zeta).$$

In both cases, the claim holds, as desired.

Now, let $\psi \in \mathcal{S}$. Then $\psi - g$ is subharmonic, and

$$\limsup_{\substack{z\to\zeta \\ z\in U}} \psi(z) - g(z) \leq 0,$$

so by the **Maximum Principle 2.3.4**, $\psi - g < 0$, so $\psi < g$ in U. So see that

$$\limsup_{\substack{z\to w \\ z\in U}} \psi(z) \leq g(w) = f(w) + \varepsilon$$

for all $\psi \in \mathcal{S}$. So

$$\limsup_{\substack{z\to w \\ z\in U}} u(z) \leq f(w) + \varepsilon$$

for any $\varepsilon > 0$, so

$$\limsup_{\substack{z\to w \\ z\in U}} u(z) \leq f(w).$$

Now, let

$$\widetilde{g}(z) = f(w) - \varepsilon + \frac{b_w(z)}{\mu}(M + f(w));$$

then $\widetilde{g} \in sh(U)$. We'll do a similar rundown. We claim that $\widetilde{g}(\zeta) < f(\zeta)$ for all $\zeta \in \partial U$.

If $\zeta \in \partial U \cap D(w,\delta)$, then $|f(\zeta) - f(w)| < \varepsilon$, so

$$f(\zeta) > f(w) - \varepsilon \geq f(w) - \varepsilon + \frac{b_w(z)}{\mu}(M + f(w)) = \widetilde{g}(\zeta).$$

If $\zeta \in \partial U \setminus D(w,\delta) \subseteq \overline{U} \setminus D(w,\delta)$, then $b_w(\zeta) < -\mu$, so

$$\widetilde{g}(\zeta) \leq f(w) - \varepsilon - M - f(w) = -M - \varepsilon < f(\zeta).$$

Therefore, $\widetilde{g} \in \mathcal{S}$ and $u(z) \geq \widetilde{g}(z)$ for all $z \in U$. Thus,

$$f(w) - \varepsilon = \lim_{\substack{z\to w \\ z\in U}} \widetilde{g}(z) \leq \liminf_{\substack{z\to w \\ z\in U}} u(z)$$

for all $\varepsilon > 0$, so

$$f(w) \leq \liminf_{\substack{z\to w \\ z\in U}} u(z) \leq \limsup_{\substack{z\to w \\ z\in U}} u(z) \leq f(w).$$

Therefore,

$$\lim_{\substack{z\to w \\ z\in U}} u(z) = f(w).$$

The proof is complete! $\square$

A closing remark: If U has barriers only at some points $P \in \partial U$, and f is continuous at some of those points, we can still construct a $u \in h(U)$ with

$$\lim_{\substack{z\to w \\ z\in U}} u(z) = f(w)$$

for all w at which f is continuous and b_w exists.

We finally turn to a previously alluded to result. We know that the **Riemann Mapping Theorem 2.2.2** says nothing about open sets that are not simply connected. We use results about harmonic functions to show that annuli are only conformally equivalent when the ratio of their radii are equal.

Theorem 2.4.22. *Let $R_1, R_2 > 1$, and set $A_1 = \{z \in \mathbf{C} \mid 1 \leq |z| \leq R_1\}$ and $A_2 = \{z \in \mathbf{C} \mid 1 \leq |z| \leq R_2\}$. Then A_1 is conformally equivalent to A_2 if and only if $R_1 = R_2$.*

Proof. Clearly, if $R_1 = R_2$, then A_1 is conformally equivalent to A_2.

For the other direction, suppse A_1 is conformally equivalent to A_2. Then there is a $\varphi : A_1 \to A_2$ such that φ is a bijection and φ is holomorphic (and thus, φ^{-1} is). If $K \subseteq A_2$ is compact, then so is $\varphi^{-1}(K)$, since φ^{-1} is continuous. It follows that if a sequence $(w_j) \subseteq A_1$ converges to ∂A_1 (that is, all accumulation points are on ∂A_1), then $(\varphi(w_j)) \subseteq A_2$ has accumulation points only on ∂A_2. Moreover, we claim that if $|w_j| \to 1$, then either for all such sequences, $|\varphi(w_j)| \to 1$, or $|\varphi(w_j)| \to R_2$. Furthermore, in the first case, if $|w_j| \to 1$ and $|w_j{}'| \to R_1$, then $|\varphi(w_j{}')| \to R_2$. In the second case, if $|w_j| \to 1$ and $|\varphi(w_j)| \to R_2$, then if $|w_j{}'| \to R_1$, $|\varphi(w_j{}')| \to 1$. We will prove this claim at the end of the proof.

Supposing these claims, after composing φ with an inversion, e.g., $R(z) = \frac{R_2}{z}$, if necessary, we may suppose that $|\varphi(z_j)| \to R_2$ as $|z_j| \to R_1$ and $|\varphi(z_j)| \to 1$ as $|z_j| \to 1$.

Consider the function

$$h(z) = \log|z| \log R_2 - \log|\varphi(z)| \log R_1.$$

Then $h \in h(A_1) \cap C(\overline{A_1})$, if we extend $h(z)$ to be 0 when $z \in \partial A_1$. By the **Maximum Principle 2.3.4** and **Minimum Principle 2.3.5**, $h \equiv 0$. From this fact, we see that $|\varphi(z)| = |z|^\beta$, where $\beta = \frac{\log R_2}{\log R_1}$.

Let $D(P, r) \subseteq A_1$. The function $F(z) = z^\beta$ can be made holomorphic on $D(P, r)$ by choosing an appropriate branch of $\log \cdot$. Also, $\frac{\varphi(z)}{F(z)} \in H(D(P, r))$, and $\left|\frac{\varphi(z)}{F(z)}\right| \equiv 1$ on $D(P, r)$. This means that $\frac{\varphi}{F} \equiv \alpha$ for some $|\alpha| = 1$, by the **Open Mapping Theorem 1.17.7**. From this, we conclude that $\varphi(z) = \alpha F(z) = \alpha z^\beta$ on $D(P, r)$. Since the computation of φ can be done on any disk $D(P, r) \subseteq A_1$ and φ is continuous, it follows that $\frac{1}{\alpha}\varphi = z^\beta$ on all of A_1.

Now, $z^\beta \in H(A_1)$ if and only if $\beta \in \mathbf{Z}$, but $\beta \in \mathbf{N}$ because $D(0, 1) \cap A_j = \emptyset$ for $j = 1$ and $j = 2$. However, φ is a biholomorphism, so $\beta = 1$. Thus, φ is just a rotation, so $R_1 = R_2$, as desired.

We now prove the claim presented at the beginning of this proof. We know that if $(w_j) \subseteq A_1$ and $|w_j| \to \partial A_1$, then $|\varphi(w_j)| \to \partial A_2$. In particular, if $\varepsilon > 0$ is small, then

$$\varphi\left(\{z \mid 1 < |z| < 1 + \varepsilon\}\right) \cap \left\{z \mid |z| = \frac{1}{2}(1 + R_2)\right\} = \emptyset.$$

For j large, $\varphi(w_j)$ must be in a fixed component of $\left\{z \in A_2 \mid |z| \neq \frac{1+R_2}{2}\right\}$. That is, $\varphi(w_j)$ cannot jump back and forth between components $\left\{z \in A_2 \mid 1 < |z| < \frac{1+R_2}{2}\right\}$ and $\left\{z \in A_2 \mid \frac{1+R_2}{2} < |z| < R_2\right\}$. The reason for this is that since

$$\varphi\left(\{z \mid 1 < |z| < 1 + \varepsilon\}\right) \cap \left\{z \mid |z| = \frac{1}{2}(1 + R_2)\right\} = \emptyset$$

and for large j_1 and j_2, w_{j_1} and w_{j_2} can be connected by a curve in $\{z \in A_1 \mid 1 < |z| < 1 + \varepsilon\}$. Hence, so can $\varphi(w_{j_1})$ and $\varphi(w_{j_2})$ in $\left\{z \in A_2 \mid |z| \neq \frac{1+R_2}{2}\right\}$.

Thus, we may conclude that

$$\lim_{j \to \infty} |\varphi(w_j)| = 1$$

for all sequences (w_j) with

$$\lim_{j \to \infty} |w_j| = 1,$$

OR

$$\lim_{j \to \infty} |\varphi(w_j)| = R_2$$

for all sequences (w_j) with

$$\lim_{j\to\infty} |w_j| = 1.$$

By composing with an inversion, we may assume the former case happens.

If we are in the case that $|w_j| \to R_1$ and $|\omega_j| \to 1$ implies $|\varphi(\omega_j)| \to 1$, then we must have that $|\varphi(w_j)| \to R_2$, or else $|\varphi(w)|$ would achieve its max on A_2 and hence be constant.

In the other case, we use an inversion to reduce it to the case above. $\square$

2.5 Infinite Series and Products

Definitions: normal convergence (of a series), uniformly Cauchy on compact sets, convergence (of a product), uniform convergence (of a product), Weierstass elementary factors, regular

Main Idea: Infinite products, on their own, often have results that follow pretty immediately from infinite sums, since log and exp take us back and forth. Where infinite products shine, however, is in factoring holomorphic and meromorphic functions. We use infinite products to factor entire functions via the Weierstrass Factorization Theorem. We also see results by Weierstrass and Mittag-Leffler that allow us to prescribe zero sets and pole sets to meromorphic functions, as long as such sets have no accumulation points on the interior of your set (since such functions must be identically 0 or ∞).

The basic beginning concept is as follows: let $(f_j) \subseteq H(U, \mathbf{C})$. We will study

$$\sum_{j=1}^{\infty} f_j \qquad \text{and} \qquad \prod_{j=1}^{\infty}(1+f_j)$$

from the partial sums and partial products

$$S_n = \sum_{j=1}^{n} f_j \qquad \text{and} \qquad P_n = \prod_{j=1}^{n}(1+f_j).$$

Definition 2.5.1. The series

$$\sum_{j=1}^{\infty} f_j$$

converges normally on U if $S_n \to f$ normally on U. In this case, the limit

$$f(z) = \sum_{j=1}^{\infty} f_j(z)$$

will be holomorphic on U when the f_j are.

Definition 2.5.2. The series

$$\sum_{j=1}^{\infty} f_j(z)$$

is said to be **uniformly Cauchy on compact sets** if for each $K \subseteq U$ and $\varepsilon > 0$, there exists $N = N(K, \varepsilon)$ such that if $m \geq n \geq N$, then

$$\left|\sum_{j=n}^{m} f_j(z)\right| < \varepsilon$$

for all $z \in K$.

For products, our goal is to develop machinery to construct holomorphic and meromorphic functions with prescribed behavior (like having a particular zero set). We start with a discussion of products of the form $\prod_{j=1}^{\infty}(1+a_j)$.

Definition 2.5.3. An infinite product

$$\prod_{j=1}^{\infty}(1+a_j)$$

is said to **converge** if

1. only a finite number of the a_js are -1, say, $a_{j_1}, ..., a_{j_k}$, and
2. if $N_0 \geq j_k + 1$, then

$$\lim_{n\to\infty} \prod_{j=N_0}^{n} (1+a_j)$$

exists, and is nonzero.

If an infinite product converges, then we define its value

$$\prod_{j=1}^{\infty}(1+a_j) = \left(\prod_{j=1}^{N_0}(1+a_j)\right) \cdot \left(\prod_{j=N_0+1}^{\infty} (1+a_j)\right).$$

Let's make a couple of remarks about this definition. First, the value of

$$\prod_{j=1}^{\infty}(1+a_j)$$

is independent of N_0, provided $N_0 > j_k$. Second, because only finitely many of the a_js may be -1, if the product converges, then

$$\lim_{m,n\to\infty} \prod_{j=n}^{m}(1+a_j) = 1.$$

Third, if the infinite product converges, then the limit of the partial products,

$$\lim_{n\to\infty} \prod_{j=1}^{n}(1+a_j),$$

exists, but the converse is false.

Example 2.5.4. If $a_j = \frac{-1}{2}$ for all j, then

$$\prod_{j=1}^{n}(1+a_j) = \frac{1}{2^n} \to 0$$

as $n \to \infty$, but

$$\prod_{j=1}^{\infty}(1+a_j)$$

diverges to 0.

Lemma 2.5.5. *If $x \in [0,1]$, then $1 + x \leq e^x \leq 1 + 2x$.*

Proof. We know that

$$\sum_{j=2}^{\infty} \frac{1}{j!} < \sum_{j=2}^{\infty} \frac{1}{2^{j-1}} = 1.$$

By a Taylor expansion argument,

$$1 + x \leq \sum_{j=0}^{\infty} \frac{x^j}{j!} \leq 1 + x + x \sum_{j=2}^{\infty} \frac{1}{j!} < 1 + 2x.$$

$\square$

Corollary 2.5.6. *If $a_j \in \mathbf{C}$, $|a_j| < 1$, then the partial products P_N for*

$$\prod_{j=1}^{\infty} (1 + |a_j|)$$

satisfy

$$\exp\left(\frac{1}{2}\sum_{j=1}^{N} |a_j|\right) \leq P_N \leq \exp\left(\sum_{j=1}^{N} |a_j|\right).$$

Proof. Since $1 + |a_j| \leq e^{|a_j|}$ by **Lemma 2.5.5**, it follows that

$$P_N = \prod_{j=1}^{N} (1 + |a_j|) \leq \prod_{j=1}^{N} e^{|a_j|} = e^{S_N}.$$

For the second inequality, we know that $1 + |a_j| = 1 + 2\left(\frac{1}{2}|a_j|\right) \geq e^{\frac{1}{2}|a_j|}$, so

$$P_N = \prod_{j=1}^{N} (1 + |a_j|) \geq \prod_{j=1}^{N} e^{\frac{1}{2}|a_j|} = e^{\frac{1}{2}S_N}.$$

$\square$

Corollary 2.5.7. *If*

$$\sum_{j=1}^{\infty} |a_j| < \infty,$$

then

$$\prod_{j=1}^{\infty} (1 + |a_j|)$$

converges.

Proof. Say

$$\sum_{j=1}^{\infty} |a_j| = M.$$

By **Corollary 2.5.6**, $P_N \leq e^M$. Also, $1 \leq P_1 \leq P_2 \leq \cdots$, so (P_N) is increasing and bounded above; hence, the limit exists. Also, $|a_j| \geq 0$, so the conditions of product convergence are satisfied automatically. $\square$

Corollary 2.5.8. *If*

$$\prod_{j=1}^{\infty}(1+|a_j|)$$

converges, then

$$\sum_{j=1}^{\infty}|a_j|$$

converges.

Proof. By **Corollary 2.5.6**, $P_N \geq e^{\frac{1}{2}S_N}$, and since P_N is bounded and S_N is monotonic, the series

$$\sum_{j=1}^{\infty}|a_j|$$

converges. $\square$

We will see that studying infinite products is, on its own, rather trivial, since, depending on branch cuts, $\exp : \mathbf{C} \to \mathbf{C} \setminus \{0\}$ and $\log : \mathbf{C} \setminus \{0\} \to \mathbf{C}$ take us back and forth between sums and products, so results we know from one generally hold in the other. Until we get to that point, however, we show these results explicitly. Showing absolute convergence of products implies convergence is next, but it takes some work, because products can diverge to 0.

Lemma 2.5.9. *Let $(a_j) \subseteq \mathbf{C}$. Set*

$$P_N = \prod_{j=1}^{N}(1+a_j),$$

and set

$$\widetilde{P}_N = \prod_{j=1}^{N}(1+|a_j|).$$

Then

$$|P_N - 1| \leq \widetilde{P}_N - 1.$$

Before the proof, note that by renumbering, **Lemma 2.5.9** also means that

$$\left| -1 + \prod_{j=N+1}^{M}(1+a_j) \right| \leq -1 + \prod_{j=N+1}^{M}(1+|a_j|).$$

Proof. By direction expansion, P_N is 1 plus the monomial terms consisting of the product of the a_js. Also, $\widetilde{P}_N$ is 1 plus the absolute values of the same monomials. By subtracting 1, the desired inequality follows from the triangle inequality. $\square$

Theorem 2.5.10. *If the infinite product*

$$\prod_{j=1}^{\infty}(1+|a_j|)$$

converges, then so does

$$\prod_{j=1}^{\infty}(1+a_j).$$

Proof. By **Corollary 2.5.8**,

$$\sum_{j=1}^{\infty} |a_j|$$

converges, since

$$\prod_{j=1}^{\infty} (1 + |a_j|)$$

does. Therefore,

$$\lim_{j \to \infty} |a_j| = 0,$$

and in particular, there exists N_0 such that if $n \geq N_0$, $a_n \neq -1$. For $J > N_0$, set

$$Q_J = \prod_{j=N_0+1}^{J} (1 + a_j),$$

and set

$$\widetilde{Q}_J = \prod_{j=N_0+1}^{J} (1 + |a_j|).$$

If $M > N > N_0$, then

$$|Q_M - Q_N| = |Q_N| \cdot \left| \prod_{j=N+1}^{M} (1 + a_j) - 1 \right| \leq |Q_N| \cdot \left(\prod_{j=N+1}^{M} (1 + |a_j|) - 1 \right),$$

by the remark following **Lemma 2.5.9**. This means that

$$|Q_M - Q_n| \leq |\widetilde{Q}_N| \cdot \left| \prod_{j=N+1}^{M} (1 + |a_j|) - 1 \right| = |\widetilde{Q}_M - \widetilde{Q}_N|.$$

The convergence of $(\widetilde{Q}_N)$ implies convergence of (Q_N). We know from the discussion after the definition of convergence of an infinite product that

$$\lim_{M,N \to \infty} \prod_{j=N+1}^{M} (1 + a_j) = 1,$$

so we can choose $N > N_0$ large enough so that

$$-1 + \prod_{j=N+1}^{M} (1 + |a_j|) < \frac{1}{2}$$

for all $M > N$. Then, by **Lemma 2.5.9**,

$$\left| -1 + \prod_{j=N+1}^{M} (1 + a_j) \right| \leq -1 + \prod_{j=N+1}^{M} (1 + |a_j|) < \frac{1}{2},$$

which means that

$$\left| \prod_{j=N}^{M} (1 + a_j) - 1 \right| > \frac{1}{2}.$$

Hence,

$$\lim_{M\to\infty}|Q_M| = \lim_{M\to\infty}\left|\prod_{j=N_0+1}^{N}(1+a_j)\right|\cdot\left|\prod_{j=N+1}^{M}(1+a_j)\right| \geq \frac{1}{2}\left|\prod_{j=N_0+1}^{N}(1+a_j)\right|.$$

Therefore,

$$\prod_{j=1}^{\infty}(1+a_j)$$

converges, as desired. □

Corollary 2.5.11. *If*

$$\sum_{j=1}^{\infty}|a_j| < \infty$$

converges, then

$$\prod_{j=1}^{\infty}(1+a_j)$$

converges as well.

Proof. First apply **Corollary 2.5.7**, then apply **Theorem 2.5.10**. □

Note that a restatement of **Corollary 2.5.11** is the following:
If

$$\sum_{j=1}^{\infty}|1-b_j| < \infty$$

converges, then

$$\prod_{j=1}^{\infty} b_j$$

converges.

Definition 2.5.12. Let (f_j) be a sequence of functions. We say that the product

$$\prod_{j=1}^{\infty}(1+f_j(z))$$

converges uniformly on a set E if

1. $$\prod_{j=1}^{\infty}(1+f_j(z))$$

   converges for each $z \in E$, and

2. the sequence

$$\left(\prod_{j=1}^{N}(1+f_j(z))\right)$$

converges uniformly on E to

$$\prod_{j=1}^{\infty}(1+f_j(z)).$$

Theorem 2.5.13. *Let $U \subseteq \mathbf{C}$ be open. Suppose $f_j \in H(U)$ for $j \in \mathbf{N}$. If*

$$\sum_{j=1}^{\infty}|f_j|$$

converges uniformly on compact sets, then

$$\prod_{j=1}^{\infty}(1+f_j(z))$$

converges uniformly on compact sets. The function vanishes at $z_0 \in U$ if and only if $f_j(z_0) = -1$ for some j. The multiplicity of the zero at z_0 is the sum of the multiplicities of the zeros of the functions $(1+f_j(z))$ at z_0.

Proof. Let $K \subseteq U$ be compact. Since

$$\sum_{j=1}^{\infty}|f_j|$$

converges uniformly on K, there exists $C > 0$ such that

$$\sum_{j=1}^{N}|f_j| \leq C$$

for all $z \in K$ and $N \in \mathbf{N}$. By **Corollary 2.5.6**,

$$\prod_{j=1}^{N}(1+|f_j|) \leq e^C$$

on K.

Let $0 < \varepsilon < 1$. Choose L large enough so that if $M \geq N \geq L$, then

$$\sum_{j=N}^{M}|f_j(z)| < \varepsilon$$

for all $z \in K$. If $M \geq N \geq L$, then by **Lemma 2.5.9** and **Corollary 2.5.6**,

$$|P_M - P_N| \leq |P_N(z)| \cdot \left|\prod_{j=N+1}^{M}(1+|f_j|) - 1\right| \leq \prod_{j=1}^{N}(1+|f_j(z)|) \cdot \left(\exp\left(\sum_{j=N+1}^{M}|f_j(z)|\right) - 1\right) \leq e^C\left(e^{\varepsilon}-1\right).$$

Since $e^{\varepsilon} \to 1$ as $\varepsilon \to 0$, it follows that $(P_N(z))$ is uniformly Cauchy.

Suppose that $P(z_0) = 0$ for some $z_0 \in U$. Then, by the definition of convergence of infinite products, there exists j_0 such that

$$\lim_{N\to\infty} \prod_{j=j_0+1}^{N} (1 + f_j(z))$$

is nonvanishing at z_0. We know from **Corollary 2.5.11** that

$$\prod_{j=1}^{\infty} (1 + f_j(z))$$

converges, so divergence to 0 is a nonissue.

By the uniform convergence already established,

$$\lim_{N\to\infty} \prod_{j=j_0+1}^{N} (1 + f_j(z))$$

is holomorphic. In particular, the limit is continuous, and therefore nonzero in a neighborhood V of z_0. Now,

$$\prod_{j=1}^{\infty} (1 + f_j(z)) = \left(\prod_{j=1}^{j_0} (1 + f_j(z)) \right) \cdot \left(\lim_{N\to\infty} \prod_{k=j_0+1}^{N} (1 + f_k(z)) \right).$$

Since the second factor is holomorphic and nonvanishing on V, the statement about the zeros of

$$\prod_{j=1}^{\infty} (1 + f_j(z))$$

and their multiplicities follows by inspection of the first factor. $\square$

We now turn towards developing the **Weierstrass Factorization Theorem 2.5.18**. Our goal is to factor holomorphic functions in a manner akin to the factorization of polynomials. Since holomorphic functions can have infinitely many zeros, the basic building blocks need to be more complicated than $(z - a_j)$ factors. Instead, we use Weierstrass elementary factors.

Definition 2.5.14. We define **Weierstass elementary factors**, $E_n(z)$, as follows: define $E_0(z) = 1 - z$, and for $n \in \mathbf{N}$, set $E_n(z) = (1 - z) \exp\left(z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots + \frac{z^n}{n} \right)$.

The key fact here is that E_n is close to 1 if $|z|$ is small. This is reasonable, since $z + \frac{z^2}{2} + \cdots + \frac{z^n}{n}$ is the nth Taylor polynomial for $-\log(1 - z)$.

Lemma 2.5.15. *If* $|z| < 1$*, then* $|1 - E_n(z)| \le |z|^{n+1}$.

Proof. The proof is via induction. If $n = 0$, then $1 - E_0(z) = z$, so the $n = 0$ case is trivial. Assume $n \ge 1$.

Now, $E_n(z) \in H(\mathbf{C})$, so we can write

$$E_n(z) = 1 + \sum_{j=1}^{\infty} b_j z^j.$$

We claim that $b_1 = \cdots = b_n = 0$, and $b_j \le 0$ if $j \ge n + 1$. To prove the claim, observe that

$$\begin{aligned} {E_n}'(z) &= -\exp\left(z + \cdots + \frac{z^n}{n} \right) + (1 - z)\left(1 + z + \cdots + z^{n-1}\right) \exp\left(z + \cdots + \frac{z^n}{n} \right) \\ &= \exp\left(z + \cdots + \frac{z^n}{n} \right) \left(-1 + 1 + z + \cdots + z^{p-1} - z - z^2 - \cdots - z^p \right) \\ &= -z^p \exp\left(z + \cdots + \frac{z^n}{n} \right), \end{aligned}$$

which is entire, so

$$-z^p \exp\left(z + \cdots + \frac{z^n}{n}\right) = -z^p \left(1 + \sum_{j=1}^{\infty} \alpha_j z^j\right).$$

By series expansion and the fact that the Taylor series for e^z centered at 0 has only positive coefficients, $\alpha_j > 0$ for all $j \in \mathbf{N}$. Thus,

$$E_n{}'(z) = -z^p + \sum_{j=n+1}^{\infty} (-\alpha_{j-n}) z^j.$$

Also,

$$E_n{}'(z) = \sum_{j=1}^{\infty} j b_j z^{j-1},\ E_n(0) = 1.$$

Comparing the coefficients verifies the claim.

Continuing,

$$0 = E_n(1) = 1 + \sum_{j=n+1}^{\infty} b_j,\ b_j \leq 0.$$

This means that

$$\sum_{j=n+1}^{\infty} |b_j| = 1.$$

We now estimate that for $|z| \leq 1$,

$$|E_n(z) - 1| = \left| \sum_{j=n+1}^{\infty} b_j z^j \right| \leq |z|^{n+1} \sum_{j=n+1}^{\infty} |b_j| = |z|^{n+1}.$$

Proof complete. □

Theorem 2.5.16. *Let (a_j) be a sequence of (not necessarily distinct) nonzero complex numbers with no accumulation point in $\mathbf{C}$. If $\{p_j\} \subseteq \mathbf{N}$ satisfies*

$$\sum_{n=1}^{\infty} \left(\frac{r}{|a_n|}\right)^{p_n+1} < \infty$$

for every $r > 0$, then the infinite product

$$\prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$$

converges uniformly on compact subsets of $\mathbf{C}$ to an entire function F. The zeros of F are precisely the points $\{a_j\}$, counted with multiplicity.

Proof. For a given $r > 0$, there exists $N \in \mathbf{N}$ such that if $n \geq N$, $|a_n| > r$ (otherwise, there would be an accumulation point). Thus, for all $n \geq N$ and $z \in \overline{D(0,r)}$, **Lemma 2.5.15** yields

$$\left| E_{p_n}\left(\frac{z}{a_n}\right) - 1 \right| \leq \left| \frac{z}{a_n} \right|^{p_n+1} \leq \left| \frac{r}{a_n} \right|^{p_n+1}.$$

Thus,

$$\sum_{n=N}^{\infty} \left| E_{p_n}\left(\frac{z}{a_n}\right) - 1 \right| < \infty.$$

Uniform convergence of

$$\sum_{n=1}^{\infty} \left(E_{p_n}\left(\frac{z}{a_n}\right) - 1 \right)$$

on $\overline{D(0,r)}$ follows by the Weierstrass M-test. Thus, the product

$$\prod_{n=1}^{\infty} \left(1 + \left(E_{p_n}\left(\frac{z}{a_n}\right) - 1\right)\right) = \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$$

converges uniformly on $\overline{D(0,r)}$, by **Theorem 2.5.13**. Since r was arbitrary, the infinite product defines a function $F \in H(\mathbf{C})$. The statement regarding the zeros of F follows immediately from **Theorem 2.5.13**. □

Corollary 2.5.17. *Let $\{a_n\} \subseteq \mathbf{C}$ have no finite accumulation point. Then there exists an entire function f with zero set precisely $\{a_n\}$, counting multiplicity.*

Proof. We may assume without loss of generality that $a_1, ..., a_m = 0$, and $a_{m+1}, a_{m+2}, ... \neq 0$.

Let $r > 0$ be fixed but arbitrary. There exists $N > m$ such that when $n \geq N$, $|a_n| > 2r$. This forces

$$\sum_{n=N}^{\infty} \frac{r}{|a_n|} < \sum_{n=N}^{\infty} \left(\frac{1}{2}\right)^n < \infty,$$

so the hypotheses of **Theorem 2.5.16** are satisfied with $p_n = n - 1$, and the entire function

$$f(z) = z^m \prod_{n=m+1}^{\infty} E_{n-1}\left(\frac{z}{a_n}\right)$$

satisfies the conclusion of the theorem. □

We have now reached the **Weierstass Factorization Theorem 2.5.18**. It lets us factor entire functions. We will later see another theorem by Weierstass (**Theorem 2.5.19**) that lets us prescribe zero sets to holomorphic functions on any open set U, as long as their zero sets are well enough behaved.

Theorem 2.5.18 (The Weierstass Factorization Theorem). *Let $f \in H(\mathbf{C})$. Suppose f vanishes to order m at 0, $m \geq 0$. Let $\{a_n\}$ be the other zeros of f, listed with multiplicities. Then there is an entire function g such that*

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_{n-1}\left(\frac{z}{a_n}\right).$$

Proof. By **Corollary 2.5.17**, the function

$$h(z) = z^m \prod_{n=1}^{\infty} E_{n-1}\left(\frac{z}{a_n}\right)$$

has the same zeros of f, counting multiplicities. Thus, by the **Riemann Removable Singularities Theorem 1.16.2**, $\frac{f}{h}$ is entire and nonvanishing. Since $\mathbf{C}$ is holomorphically simply connected, by **Lemma 2.2.13**, $\frac{f}{h}$ has a holomorphic logarithm g on $\mathbf{C}$; thus, $f = e^g h$, as desired. □

Here is the aforementioned theorem by Weierstrass that lets us prescribe zero sets on any open set.

Theorem 2.5.19 (**Weierstrass**). *Let $U \subseteq \mathbf{C}$ be any open set. Let $a_1, a_2, \ldots$ be a finite or infinite sequence in U, possibly with repetition, with no accumulation points in U. Then there exists $f \in H(U)$ whose zero set is precisely $\{a_n\}$, counting multiplicity.*

Proof. If $\{a_n\}$ is a finite set, then the polynomial

$$P(z) = \prod_{j=1}^{N} (z - a_j)$$

satisfies the requirements. Thus, we may assume $\{a_n\}$ is infinite.

We view U as a subset of the Riemann sphere $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. Choose $q \in U$ so that $q \notin \{a_n\}$. Next, we apply the fractional linear transformation $z \mapsto \frac{1}{z-q}$, and solve the problem on $\Omega = \left\{ w \in \widehat{\mathbf{C}} \mid w = \frac{1}{z-q} \text{ for } z \in U \right\}$. A solution on Ω immediately produces a solution on U by the inverse of $\frac{1}{z-q}$; $w = \frac{1}{z-q}$ if and only if $z = \frac{1+qw}{w}$.

Ω is unbounded and ∂U gets mapped to a bounded set, and $d(\partial U, q) \geq c > 0$. Additionally, where $\widetilde{a_n} = \frac{1}{a_n - q}$,

1. $\Omega \subsetneq \widehat{\mathbf{C}}$,
2. $\widehat{\mathbf{C}} \setminus \Omega$ is compact in $\mathbf{C}$,
3. $\{\widetilde{a_n}\} \cup \{\infty\} \subseteq \Omega$, and
4. $\{\infty\} \cap \{\widetilde{a_n}\} = \emptyset$.

By hypothesis, the accumulation points of $\{\widetilde{a_n}\}$ are all in $\partial\Omega$, hence any compact subset of Ω contains only finitely many $\widetilde{a_n}$s. By 2. and the fact that $d(\bullet, \widehat{\mathbf{C}} \setminus \Omega)$ is continuous, for each $\widetilde{a_n} \in \Omega$ there is a (not necessarily unique) point $\widehat{a_n} \in \widehat{\mathbf{C}} \setminus \Omega$ of minimial distance from $\widetilde{a_n}$.

Let $K \subseteq \Omega$ be compact. Then there exists $\delta > 0$ such that $d(K, \widehat{\mathbf{C}} \setminus \Omega) \geq \delta > 0$, which means that if $d_n = |\widetilde{a_n} - \widehat{a_n}|$, then $d_n \to 0$ as $n \to \infty$, and if $z \in K$ and $w \in \widehat{\mathbf{C}} \setminus \Omega$, then $|z - w| \geq \delta > 0$. In particular, $|z - \widehat{a_n}| \geq \delta$ for all $z \in K$ and all $n \in \mathbf{N}$.

Since $d_n \to 0$, there exists n_0 so that if $n \geq n_0$, then $d_n < \frac{1}{2}|z - \widehat{a_n}|$ for all $z \in K$. In other words,

$$\left| \frac{\widetilde{a_n} - \widehat{a_n}}{z - \widehat{a_n}} \right| = \frac{d_n}{|z - \widehat{a_n}|} < \frac{\frac{1}{2}|z - \widehat{a_n}|}{|z - \widehat{a_n}|} = \frac{1}{2}.$$

We may apply **Theorem 2.5.13** and **Lemma 2.5.15** with z replaced by $\frac{\widetilde{a_n} - \widehat{a_n}}{z - \widehat{a_n}}$ so that

$$f(z) = \prod_{n=1}^{\infty} E_n \left(\frac{\widetilde{a_n} - \widehat{a_n}}{z - \widehat{a_n}} \right)$$

converges uniformly on K. Since K was arbitrary, $f \in H(\Omega)$ satisfies the conclusion. $\square$

We will soon need the following geometric construction:

Lemma 2.5.20. *Let $U \subsetneq \mathbf{C}$ be open. There exists a countable set $A \subseteq U$ such that*

1. *A has no accumulation point in U, and*
2. *every $P \in \partial U$ is an accumulation point of A; i.e., $A' = \partial U$.*

Proof. Since $\mathbf{C} \setminus U \neq \emptyset$, the function

$$d(z) = \inf_{w \in \mathbf{C} \setminus U} |z - w| = d(z, \mathbf{C} \setminus U)$$

is well-defined, positive, and finite on U.

For $j \in \mathbf{N}$, set

$$A_j = \left\{ z \in U \mid \frac{1}{4^{j+1}} < d(z) \leq \frac{1}{4^j} \right\}.$$

For each j, set Q_j to be the collection of boxes with vertices of the form $\left(\frac{k}{8^j}, \frac{\ell}{8^j}\right)$, $\left(\frac{k+1}{8^j}, \frac{\ell}{8^j}\right)$, $\left(\frac{k}{8^j}, \frac{\ell+1}{8^j}\right)$, and $\left(\frac{k+1}{8^j}, \frac{\ell+1}{8^j}\right)$. Q_j is countable, so we may enumerate them and write $Q_j = \{q_{j,p}\}_{p=1}^{\infty}$.

Next, for each $j \in \mathbf{N}$ and each $q_{j,p}$ with closure intersecting A_j, the center of $q_{j,p}$ lies in U by the triangle inequality. Namely, the distance between any point in $\overline{q_{j,p}}$ and $\mathbf{C} \setminus U$ is greater than or equal to $\frac{1}{4^{j+1}}$, while the distance between any point in $\overline{q_{j,p}}$ and the center is less than $\frac{2}{8^j}$. By a similar argument, the distance from the center of $q_{j,p}$ to $\mathbf{C} \setminus U$ is less than or equal to $\frac{1}{4^{j-1}}$.

Let $P_{j,p}$ be the center of $q_{j,p}$ when $\overline{q_{j,p}} \cap A_j = \emptyset$. Let $D_j = \{P_{j,p}\}$. Then D_j is either finite or countable.

Now, set

$$A = \bigcup_{j=1}^{\infty} D_j.$$

Then A is countable. We claim that this A works; i.e., we need to show that A has no accumulation points in U and that $A' = \partial U$.

For 1., observe that if $a, a' \in A$ and $|a - a'| < \frac{1}{16^{j_0}}$, then $a, a' \in \bigcup_{j \geq j_0} D_j$. But this means that $d(a) < \frac{1}{4^{j_0 - 1}}$ and $d(a') < \frac{1}{4^{j_0-1}}$, a contradiction.

To see 2., let $P \in \partial U$ and $\varepsilon > 0$. Without loss of generality, assume $\varepsilon < \frac{1}{4}$. Since $P \in \partial U$, $D(P, \varepsilon) \cap U \neq \emptyset$, since U is open. Say $z \in D(P, \varepsilon) \cap U$. Then $d(z) < \varepsilon < \frac{1}{4}$. Choose $j \in \mathbf{N}$ so that $\frac{1}{4^{j+1}} < d(z) \leq \frac{1}{4^j}$. Then $z \in A_j$, so there exists $q_{j,p}$ with $z \in \overline{q_{j,p}}$. This means that $P_{j,p} \in D_j \subseteq A$ and $d(P, P_{j,p}) \leq d(P, z) + d(z, P_{j,p}) < \varepsilon + \frac{2}{8^j} \leq 2\varepsilon$.

The proof is complete. $\square$

Lemma 2.5.20 has major implications for the domain of existence of holomorphic functions.

Definition 2.5.21. Let $U \subsetneq \mathbf{C}$ be open, and let $P \in \partial U$. P is said to be **regular** for $f \in H(U)$ if there exists $r > 0$ and $\widetilde{f} \in H(D(P, r))$ such that $\widetilde{f}\Big|_{D(P,r) \cap U} = f\Big|_{D(P,r) \cap U}$. In other words, $\widetilde{f}$ is a holomorphic extension of f to $D(P, r) \cup U$.

Corollary 2.5.22. *Let $U \subsetneq \mathbf{C}$ be a connected, open set. Then there exists $f \in H(U)$ for which no point $P \in \partial U$ is regular.*

Proof. Let $A \subseteq U$ satisfy the conclusion of **Lemma 2.5.20**. Since A has no accumulation points in U, we may apply **Theorem 2.5.19** to establish the existence of a holomorphic function $f \in H(U)$ with the zero set of f equal to A.

If $P \in \partial U$ were regular for f, then there would be a disk $D(P, r)$ and $\widetilde{f} \in H(D(P, r))$ with $\widetilde{f}\Big|_{D(P,r) \cap U} = f\Big|_{D(P,r) \cap U}$. By the construction of A, P is an accumulation point of A, since $P \in \partial U$. This is a contradiction of the fact that the zero set of $\widetilde{f}$ restricted to $D(P, r) \cap U$ equals $A \cap D(P, r) \cap U$ (see **Theorem 1.13.2**). $\square$

Corollary 2.5.23. *Let $U \subseteq \mathbf{C}$ be open. Let m be meromorphic on U. Then, there exists $f, g \in H(U)$ so that $m(z) = \frac{f(z)}{g(z)}$.*

Proof. Let $a_1, \ldots$ be the poles of m, listed with multiplicity. By **Theorem 2.5.19**, there exists a function $g \in H(U)$ with the zero set of g exactly $\{a_1, \ldots\}$. The **Riemann Removable Singularities Theorem 1.16.2** implies $f = mg \in H(U)$. Thus, $m = \frac{f}{g}$. $\square$

In addition to prescribing zeros, we can also prescribe poles. Certainly, if we can prescribe the zero set of f, then we have prescribed the pole set of $\frac{1}{f}$, but we can do better; we can actually prescribe the entire negatively indexed Laurent expansion. First, a useful lemma:

Lemma 2.5.24 (The "Pole-Pushing Lemma"). *Let $\alpha, \beta \in \mathbf{C}$. Define*

$$A(z) = \sum_{j=-M}^{-1} a_j (z-\alpha)^j$$

for some $M \geq 1$ and $a_{-M}, ..., a_{-1} \in \mathbf{C}$. Fix $r > |\alpha - \beta|$, and let $\varepsilon > 0$. Then, there exists a finite Laurent expansion,

$$B(z) = \sum_{j=-N}^{K} b_j (z-\beta)^j,$$

for some $N \geq 1$, $K \geq -1$, such that $|A(z) - B(z)| < \varepsilon$ for all $z \in \widehat{\mathbf{C}} \setminus D(\beta, r)$.

Essentially, we can push poles around.

Proof. The Laurent expansion of A about β converges uniformly to A on every set of the form $\widehat{\mathbf{C}} \setminus D(\beta, r)$ for $r > |\alpha - \beta|$. Take B to be a sufficiently large partial sum of the expansion. You're done!

Note: a more direct proof could be done via computation. See that

$$\frac{1}{z-\alpha} = \frac{1}{z-\beta} \cdot \frac{1}{1 - \frac{\alpha-\beta}{z-\beta}} = \frac{1}{z-\beta} \sum_{j=0}^{\infty} \left(\frac{\alpha - \beta}{z - \beta} \right)^j,$$

where the sum converges uniformly on $\widehat{\mathbf{C}} \setminus D(\beta, r)$. $\square$

Here is the result about prescribing poles.

Theorem 2.5.25 (Mittag-Leffler). *Let $\Omega \subseteq \mathbf{C}$ be open. Let $\{\alpha_j\}$ be a finite or countable set of distinct points of Ω with no accumulation point in Ω. Suppose that for each j, Ω_j is a neighborhood of α_j and such that $\alpha_k \notin \Omega_j$ if $j \neq k$. Further suppose that for each j, m_j is a meromorphic function on Ω_j with a pole at α_j and no other poles. Then, there exists a meromorphic function m on Ω such that $m - m_j \in H(\Omega_j)$ for every j and which has no other poles outside of $\{\alpha_j\}$.*

Note that **Theorem 2.5.25** is equivalent to the following theorem, which we prove:

Theorem 2.5.26 (Mittag-Leffler, version 2). *Let $\Omega \subseteq \mathbf{C}$ be open, and let $\{\alpha_j\}$ be a finite or countable set of distinct points in Ω with no accumulation point in Ω. Let s_j be a sequence of Laurent polynomials (or, principal parts (**principal part 1.16.16**))*

$$s_j(z) = \sum_{\ell=P(j)}^{-1} a_{\ell,j} (z - \alpha_j)^\ell.$$

Then there is a unique meromorphic function on Ω whose principal part at α_j is s_j, and which has no other poles in Ω.

Proof. If $|\{\alpha_j\}| = n$, then

$$m(z) = \sum_{j=1}^{n} s_j(z)$$

works. Thus, we may assume that $|\{\alpha_j\}| = \infty$.

As in the proof of **Theorem 2.5.19**, we may assume that $\Omega \subsetneq \widehat{\mathbf{C}}$, and that $\infty \in \Omega$.

For each j, let $\widehat{\alpha_j} \in \widehat{\mathbf{C}} \setminus \Omega$ be a (not necessarily unique) nearest point to α_j. Set $d_j = |\widehat{\alpha_j} - \alpha_j|$.

For each j, use the **Pole-Pushing Lemma 2.5.24** to find a polynomial in negative powers of $z - \widehat{\alpha_j}$, $t_j(z)$, such that $|s_j(z) - t_j(z)| < \frac{1}{2^j}$ for $z \in \widehat{\mathbf{C}} \setminus D(\widehat{\alpha_j}, 2d_j)$.

We claim that

$$m(z) = \sum_{j=1}^{\infty} (s_j(z) - t_j(z))$$

is the desired meromorphic function.

To prove this, since Ω contains a neighborhood of ∞, $\partial\Omega$ is a bounded set. Moreover, since $\{\alpha_j\}$ has no accumulation point in Ω, $d_j \to 0$ as $j \to \infty$. Fix a closed disk $\overline{D(a,r)} \subseteq \Omega \setminus \{\alpha_j\}$. Choose $J \in \mathbf{N}$ large enough that $j \geq J$ implies that $2d_j < d(\overline{D(a,r)}, \widehat{\mathbf{C}} \setminus \Omega)$. Then, for $j \geq J$, $|s_j(z) - t_j(z)| < \frac{1}{2^j}$ for $z \in \overline{D(a,r)}$. By the Weierstrass M-test, $m(z)$ converges uniformly on $\overline{D(a,r)}$. Since $\overline{D(a,r)} \subseteq \Omega \setminus \{\alpha_j\}$ was arbitrary, the proof is complete. $\square$

Lemma 2.5.27. *Let $\Omega \subseteq \mathbf{C}$ be open. Let $\alpha \in \Omega$, and let $e_0, ..., e_p \in \mathbf{C}$ be given. Suppose that $g \in H(\Omega)$ has a zero of order $p+1$ at α. Then, there exists a Laurent polynomial*

$$v(z) = \sum_{j=-p-1}^{-1} b_{-j}(z-\alpha)^j$$

such that

$$v(z)g(z) = e_0 + e_1(z-\alpha) + \cdots + e_p(z-\alpha)^p + \textit{higher order terms.}$$

Proof. By translation, we may assume without loss of generality that $\alpha = 0$. Then,

$$g(z) = c_{p+1}z^{p+1} + c_{p+2}z^{p+2} + \cdots$$

for small values of z. Similarly,

$$v(z) = b_1 z^{-1} + \cdots + b_{p+1}z^{-p-1}.$$

We must solve for the b_js, subject to

$$v(z)g(z) = e_0 + e_1 z + \cdots + e_p z^p + \text{higher order terms.}$$

This means that

$$\begin{aligned} v(z)g(z) = {} & b_{p+1}c_{p+1} \\ & + (b_{p+1}c_{p+2} + b_p c_{p+1})\, z \\ & + (b_{p+1}c_{p+3} + b_p c_{p+2} + b_{p-1}c_{p+1})\, z^2 \\ & + \cdots \\ & + (b_{p+1}c_{2p+1} + b_p c_{2p} + \cdots + b_1 c_{p+1})\, z^p \\ & + \cdots . \end{aligned}$$

Then, equating coefficients, this means that we need $\{b_j\}$ to satisfy:

$$\begin{aligned} b_{p+1}c_{p+1} &= e_0 \\ b_{p+1}c_{p+2} + b_p c_{p+1} &= e_1 \\ b_{p+1}c_{p+3} + b_p c_{p+2} + b_{p-1}c_{p+1} &= e_2 \\ &\vdots \\ b_{p+1}c_{2p+1} + b_p c_{2p} + \cdots + b_1 c_{p+1} &= e_p. \end{aligned}$$

These equations can be solved in succession for $b_{p+1}, ..., b_1$. $\square$

Theorem 2.5.28. *Let $\Omega \subseteq \mathbf{C}$ be open, and let $\alpha_1, \alpha_2, ...$ be a finite or countable sequence of distinct points in Ω having no interior accumulation point. For each j, let there be a given expression*

$$s_j(z) = \sum_{\ell=-M(j)}^{N(j)} a_{\ell,j}(z-\alpha_j)^\ell$$

with $M(j), N(j) \geq 0$. Then there is a meromorphic function m on Ω, holomorphic on $\Omega \setminus \{\alpha_1, \alpha_2, ...\}$, such that if $-M(j) \leq \ell \leq N(j)$, then the ℓth Laurent coefficient of m at α_j is $a_{\ell,j}$.

Proof. By **Theorem 2.5.19**, there is $h \in H(\Omega)$ with a zero of order $M(j)$ at α_j and no others. Let

$$\widetilde{s_j}(z) = h(z)s_j(z),$$

and let

$$\widehat{s_j}(z) = \sum_{\ell=0}^{N(j)+M(j)} \sigma_{\ell,j}(z-\alpha_j)^\ell$$

be the $N(j)+M(j)$ order Taylor polynomial of $\widetilde{s_j}$ about α_j.

Again by **Theorem 2.5.19**, there exists $g \in H(\Omega)$ with zeros of order $M(j)+N(j)+1$ at α_j for all j, and no other zeros.

By **Lemma 2.5.27**, there are functions

$$v_j(z) = \sum_{\ell=-N(j)-M(j)-1}^{0} b_{\ell,j}(z-\alpha_j)^\ell$$

such that

$$v_j(z)g(z) = \widehat{s_j}(z) + \text{higher order terms}.$$

Next, **Theorem 2.5.26** produces a meromorphic function $k(z)$ on Ω such that k has principal part $v_j(z)$ at α_j for all j.

But then, $g(z)k(z)$ will have no poles, and by the fact that

$$v_j(z)g(z) = \widehat{s_j}(z) + \text{higher order terms},$$

$g(z)k(z)$ will have $N(j)+M(j)$ order Taylor polynomial at α_j equal to $\widehat{s_j}(z)$ for each j.

Therefore, the meromorphic function

$$\frac{g(z)k(z)}{h(z)}$$

will satisfy the conclusion of the theorem. □

2.6 Topology (hip hop air horns)

Definitions: fundamental group, simply connected, homologous to 0, homologically trivial
Main Idea: We define the fundamental group and gloss over some powerful functorial theorems proven in detail in topology. We also prove a Cauchy Integral Theorem and Cauchy Integral Formula for connected, open sets that are not simply connected. We finally finish proving the Riemann Mapping Theorem, and then we show that simply connected sets have connected complements.

In this section, we investigate the tools to prove some results by Cauchy for non-simply connected, connected, open sets, and the remaining piece of the **Riemann Mapping Theorem 2.2.2**, the proof of **Part I**.

As a preliminary setup, fix $P \in \Omega$, a connected, open set. Let

$$\mathcal{C} = \mathcal{C}(\Omega) = \{\gamma \mid \gamma \in C([0,1], \Omega) \text{ and } \gamma(0) = \gamma(1) = P\}.$$

We would like to understand the relationship of γ_1 and γ_2 in $\mathcal{C}$ when γ_1 and γ_2 are homotopic at P to each other. Recall what it means for two loops to be homotopic at P in **homotopic as closed curves 1.9.14**; namely, there exists a homotopy $H : [0,1] \times [0,1] \to \Omega$ such that

1. $H(0,t) = \gamma_1(t)$,
2. $H(1,t) = \gamma_2(t)$, and
3. $H(s,0) = H(s,1) = P$.

Note that the property of being homotopic is an equivalence relation on $\mathcal{C}$. Let the collection of equivalence classes be called $\mathcal{H}$. We can define a binary operation on $\mathcal{H}$ that turns it into a group: namely, suppose that $\mu, \gamma \in \mathcal{C}$. We want to define $\gamma \cdot \mu$ to be the curve in $\mathcal{C}$ where the curve is γ, followed by μ. We can parameterize $\gamma \cdot \mu$ by

$$(\gamma \cdot \mu)(t) = \begin{cases} \gamma(2t) & \text{if } 0 \le t \le \frac{1}{2}; \\ \mu(2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Since $\gamma(1) = \mu(0) = P$, $\gamma \cdot \mu \in \mathcal{C}$. One can check that $\cdot$ turns $\mathcal{H}$ into a group; first, see that $\cdot$ is well-defined on equivalence classes. Then $\cdot$ is associative on equivalence classes; i.e., $(\gamma \cdot \mu) \cdot \delta = \gamma \cdot (\mu \cdot \delta)$. Further, every group has an identity element, and we declare the identity element to be the equivalence class of e, where e is the curve $e(t) = P$ for all t. If $\mu \in \mathcal{C}$, one can check that $[e] \cdot [\mu] = [\mu] \cdot [e] = [\mu]$. Finally, if $\gamma \in \mathcal{C}$, define γ^{-1} by $\gamma^{-1}(t) = \gamma(1-t)$. Both curves have the same image; γ^{-1} is simply running γ backwards. One can also check that $[\gamma^{-1}] = [\gamma]^{-1}$. Thus, the inverse is well-defined on equivalence classes.

Definition 2.6.1. The group $(\mathcal{H}, \cdot)$ is usually denoted $\pi_1(\Omega)$, and called the (first) **fundamental group**.

We remark that $\pi_1(\Omega)$ is independent, up to group isomorphism, of the base point P. Even stronger, let $\Phi : \Omega \to V$ be a continuous mapping and $\Phi(P) = Q$. Then Φ induces a mapping $\Phi_* : \pi_1(\Omega) \to \pi_1(V)$ where $\pi_1(\Omega)$ is based at P and $\pi_1(V)$ is based at Q. If $\gamma : [0,1] \to \Omega$, $[\gamma] \xmapsto{\Phi_*} [\Phi \circ \gamma]$, where $\Phi \circ \gamma : [0,1] \to V$. Moreover, if $\Phi : \Omega \to V$ and $\Psi : V \to W$, then $(\Psi \circ \Phi)_* = \Psi_* \circ \Phi_*$. Φ_* is a group homomorphism; i.e., $\Phi_*([\gamma] \cdot [\mu]) = \Phi_*([\gamma]) \cdot \Phi_*([\mu])$. Also, if it turns out that Φ is a homeomorphism, then Φ^{-1} also induces a map of homotopy groups, and it follows that Φ_* is a group isomorphismof homotopy groups.

(Note that all of this is explored in specific detail in the topology notes!)

Definition 2.6.2. A connected, open set whose fundamental group consists of one element is called **simply connected**. If a connected, open set U has $|\pi_1(U)| \ge 2$, then the book says U is multiply connected. (I will avoid this term, as it could get confused with n-connected from topology. It is simpler to say that U is not simply connected.)

We remark that if U and V are connected, open sets, where U is simply connected and V is not, then U and V are not homeomorphic. Indeed, if such a homeomorphism existed, say Φ, then the push forward Φ_* would be a group isomorphism of $\pi_1(U)$ onto $\pi_1(V)$, but $|\pi_1(U)| = 1$ while $|\pi_1(V)| \ge 2$, a contradiction.

Example 2.6.3. Let $U = D(0,1)$. We claim that U is simply connected. To see this, let $P = 0$ be the base point. If $\gamma(t) = (\gamma_1(t), \gamma_2(t)) \in \mathcal{C}(U)$ is arbitrary, then the function

$$H(s,t) = ((1-s)\gamma_1(t), (1-s)\gamma_2(t))$$

is a homotopy of γ with $\widetilde{\gamma}(t) = P$. Thus, $\mathcal{H}$ has one equivalence class.

Example 2.6.4. In contrast, let $V = \{z \mid 1 < |z| < 3\}$. Let the base point be $Q = 2$. Let us investigate the mappings $\gamma_j : [0,1] \to V$ defined by $\gamma_j(t) = 2e^{2\pi i j t}$ for $j \in \mathbf{Z}$. We will see that $[\gamma_j] \ne [\gamma_k]$ if $j \ne k$. In particular, we will see that if $[\gamma_j \cdot {\gamma_k}^{-1}] = [e]$, then by **Corollary 2.6.9**,

$$\oint_{\gamma_j \cdot {\gamma_k}^{-1}} \frac{1}{z} dz = 0,$$

while direct calculation shows now that

$$\oint_{\gamma_j \cdot \gamma_k{}^{-1}} \frac{1}{z} dz = 2\pi i(j-k).$$

Therefore, $[\gamma_j \cdot \gamma_k{}^{-1}] = [e]$ if and only if $j = k$. This means that there are at least countably many homotopy classes. In fact, $\pi_1(V) \cong (\mathbf{Z}, +)$. It will require some work, however, to establish that every closed curve with base point Q is homotopic to γ_j for some j.

We describe how such a homotopy can be built; explicit construction is up to the ambitious reader. The homotopy is from μ an arbitrary loop based at Q to γ_j for some j (with image $\gamma_j([0,1]) \subseteq \{z \mid |z| = 2\}$). In polar coordinates, consider the homotopy which is μ at time 0 and linear in r over time $[0,1]$ with radius 2 at time 1. This is continuous in time and in every path, as μ and γ_j are continuous, and by construction μ at time 0 and γ_j at time 1. Further, the homotopy always remains in V, as the radius of any point either decreases linearly down from at most 3 to 2 or increases linearly up from at least 1 to 2.

We now turn towards proving a **Cauchy Integral Theorem for Non-Simply Connected, Connected, Open Sets 2.6.8**. To establish such a formula, we need to see how to compute a line integral over a continuous path- we have only ever done so over piecewise C^1 curves. Specifically, we want to define

$$\oint_\gamma f dz,$$

where $\gamma[a,b] \to \mathbf{C}$ is a continuous curve and f is holomorphic on a neighborhood of $\gamma([a,b])$.

If $a = a_1 < a_2 < \cdots < a_k < a_{k+1} = b$ is a partition of $[a,b]$ and $\gamma_j = \gamma|_{[a_j,a_{j+1}]}$, then it must be the case that

$$\oint_\gamma f dz = \sum_{j=1}^{k} \oint_{\gamma_j} f dz.$$

Suppose further that for $j \in \{1, ..., k\}$, $\text{Range}(\gamma_j) \subseteq D_j$, where D_j is an open disk on which f is defined and holomorphic. Then f has a holomorphic antiderivative on D_j, by **Theorem 1.5.15**. Call it F_j. We would like to see that

$$\oint_{\gamma_j} f dz = F_j(\gamma_j(a_{j+1})) - F_j(\gamma_j(a_j)).$$

One can check that this sort of partition always exists, and that the resulting definition of

$$\oint_\gamma f dz$$

is independent of the subdivision. Thus, we can now integrate along continuous curves, and we can discuss the notion of the index of a closed curve that is merely continuous. The reason we extend our notions to continuous curves is that homotopies are typically (no more than) continuous, and restricting the class of homotopies to piecewise C^1 would be awkward.

Definition 2.6.5. Let $\Omega \subseteq \mathbf{C}$ be a connected, open set. Let $\gamma : [0,1] \to \Omega$ be a continuous, closed curve. We say that γ is **homologous to** 0 if $\text{Ind}_\gamma(P) = 0$ for all $P \in \mathbf{C} \setminus \Omega$.

The idea behind this definition is the following:

Suppose γ is a simple closed curve in Ω. The complement of γ has two componenets, the "interior" and "exterior." If γ is a simple closed curve that encircles a hole in Ω, then γ is not homologous to 0.

Definition 2.6.6. A connected, open set Ω is **homologically trivial** if every closed curve is homologous to 0.

We remark that a homotopically trivial curve is homologous to 0; we don't wind around *any* point, let alone those outside our set.

Lemma 2.6.7. *Let $\Omega \subseteq \mathbf{C}$ be a connected, open set, and let γ be a closed curve in Ω based at $P \in \Omega$ that is homotopic to the constant curve at P. Then γ is homologous to 0. In particular, if Ω is simply connected, then it is homologically trivial.*

Proof. Let $H(s,t)$ be a homotopy of γ to the point $P \in \Omega$. Set $H_s(t) = H(s,t)$. Let $c \in \mathbf{C} \setminus \Omega$. By definition,

$$\mathrm{Ind}_\gamma(c) = \frac{1}{2\pi i} \oint_\gamma \frac{1}{\zeta - c} d\zeta.$$

Rewrite this as

$$I_0 = \frac{1}{2\pi i} \oint_{H_0} \frac{1}{\zeta - c} d\zeta,$$

and set

$$I_s = \frac{1}{2\pi i} \oint_{H_s} \frac{1}{\zeta - c} d\zeta.$$

The continuity of H in s forces I_s to be continuous in s. Since I_s is also integer valued, I_s must be constant. When s is sufficiently close to 1, the curve H_s will be contained in a small open disk $D(P,\varepsilon)$ contained in Ω, but $c \notin D(P,\varepsilon)$, so it follows that $I_s = 0$. Thus, $I_0 = 0$ as well, so γ is homologous to 0. ☐

Theorem 2.6.8 (The Cauchy Integral Theorem for Non-Simply Connected, Connected, Open Sets). *Let $\Omega \subseteq \mathbf{C}$ be a connected, open set, and let $f \in H(\Omega)$. Then*

$$\oint_\gamma f(z) dz = 0$$

for any curve γ in Ω that is homologous to 0.

Proof. We first consider the case when Ω is bounded. Let γ be a curve that lies in Ω. Let $\mu = d(\gamma, \mathbf{C} \setminus \Omega)$, so $\mu > 0$. Let $0 < \delta < \frac{\mu}{2}$. Cover the plane with closed squares with sides parallel to the axes, disjoint interiors, and side lengths δ. Let $\{Q_j\}_{j=1}^k$ be the closed squares from the cover that lie entirely in Ω. Let

$$\Omega_\delta = \mathrm{Int}\left(\bigcup_{j=1}^k Q_j \right).$$

From the choice of δ, $\gamma \subseteq \Omega_\delta$. Set $C_\delta = \partial \Omega_\delta$. We orient C_δ as follows.

Give each Q_j the counterclockwise orientation. When two sides of the Q_j meet, the integration of f along the common side cancels. Thus, only the edges that comprise C_δ remain, and their orientations are consistent. Take that as the orientation of C_δ, so integration over C_δ is defined.

Let $c \in \Omega \setminus \Omega_\delta$. Then c lies in a square Q and $Q \neq Q_j$ for any $j \in \{1, ..., k\}$. Also, there exists $x \in Q$ such that $x \notin \Omega$. The line segment from x to c lies entirely in Q and does not intersect Ω_δ. By hypothesis, $\mathrm{Ind}_\gamma(x) = 0$. By the continuity of the integral, $\mathrm{Ind}_\gamma(c) = 0$. The element c was arbitrarily chosen from $\Omega \setminus \Omega_\delta$, so, in particular, $\mathrm{Ind}_\gamma(c) = 0$ for every $c \in C_\delta$.

Now suppose that $z \in \Omega_\delta$ and $z \in \mathrm{Int}(Q_{j_0})$. The boundary case $z \in \partial Q_{j_0}$ will follow from the continuity of the integral. By the **Cauchy Integral Formula 1.9.3** for a square, it follows that

$$\frac{1}{2\pi i} \oint_{\partial Q_j} \frac{f(\zeta)}{\zeta - z} d\zeta = \begin{cases} f(z) & \text{if } j = j_0; \\ 0 & \text{if } j \neq j_0. \end{cases}$$

By summing over j and recognizing that if $\Gamma_{j,\delta}$ is a curve that parameterizes ∂Q_j, then

$$C_\delta = \sum_{j=1}^k \Gamma_{j,\delta}.$$

Thus, by the discussion above,

$$\frac{1}{2\pi i}\oint_{C_\delta}\frac{f(\zeta)}{\zeta-z}d\zeta=\sum_{j=1}^{k}\frac{1}{2\pi i}\oint_{\Gamma_{j,\delta}}\frac{f(\zeta)}{\zeta-z}d\zeta=f(z),$$

as z is in exactly one square.

Now, integrate both sides in a complex line integral over $z\in\gamma$. Then

$$\oint_\gamma\left(\frac{1}{2\pi i}\oint_{C_\delta}\frac{f(\zeta)}{\zeta-z}d\zeta\right)dz=\oint_\gamma f(z)dz.$$

By Fubini's Theorem (which is okay, since $d(\gamma,C_\delta)>0$),

$$\oint_{C_\delta}f(\zeta)\left(\frac{1}{2\pi i}\oint_\gamma\frac{1}{\zeta-z}dz\right)d\zeta=\oint_\gamma f(z)dz.$$

Since $C_\delta\subseteq\Omega\setminus\Omega_\delta$,

$$\frac{1}{2\pi i}\oint_\gamma\frac{1}{\zeta-z}dz=-\operatorname{Ind}_\gamma(\zeta)=0$$

for $\zeta\in C_\delta$, since γ is homologous to 0. For bounded domains, the theorem is proven.

If Ω is unbounded, then simply take $\Omega'=\Omega\cap D(0,R)$ where R is large enough so that $\gamma\subseteq D(0,R)$. The proof follows as before, after observing that $\operatorname{Ind}_\gamma(z)=0$ for $z\notin D(0,R)$; hence $\operatorname{Ind}_\gamma(z)=0$ for all $z\in\mathbf{C}\setminus(\Omega\cap D(0,R))$. □

Corollary 2.6.9. *If $f\in H(\Omega)$ and $\gamma:[0,1]\to\Omega$ is a closed curve based at $P\in\Omega$ that is homotopic to a constant curve at P, then*

$$\oint_\gamma f(z)dz=0.$$

Proof. Combine **Lemma 2.6.7** and the **Cauchy Integral Theorem for Non-Simply Connected, Connected, Open Sets 2.6.8**. □

Theorem 2.6.10 (**The Cauchy Integral Formula for Non-Simply Connected, Connected, Open Sets**). *Let $\Omega\subseteq\mathbf{C}$ be a connected, open set. Let γ be a closed curve in Ω that is homologous to 0. If $z\in\Omega$ and $f\in H(\Omega)$, then*

$$\operatorname{Ind}_\gamma(z)f(z)=\frac{1}{2\pi i}\oint_\gamma\frac{f(\zeta)}{\zeta-z}d\zeta.$$

In particular, the formula holds for any closed curve that is homotopic to a point.

Proof. The proof of the first statement follows from the **Cauchy Integral Theorem for Non-Simply Connected, Connected, Open Sets 2.6.8** applied to the holomorphic function

$$F(\zeta)=\frac{f(\zeta)-f(z)}{\zeta-z}.$$

The second statement follows from **Lemma 2.6.7**. □

Now, with the goal of proving **Part I** of the **Riemann Mapping Theorem 2.2.2**, we explore the connection between **holomorphic simple connectivity 1.16.20** and **simple connectivity 1.9.15, 2.6.2**. Earlier (the **analytic version of the Riemann Mapping Theorem 2.2.12**), we observed that if $\Omega\subsetneq\mathbf{C}$ is holomorphically simply connected, then Ω is conformally equivalent to the unit disk. The goal now is to show that holomorphic simple connectivity is equivalent to (topological) simple connectivity in $\mathbf{C}$.

Lemma 2.6.11. *Holomorphic simple connectivity implies topological simple connectivity.*

Proof. If $\Omega = \mathbf{C}$, then there is nothing to show. Assume that Ω is holomorphically simply connected and $\Omega \neq \mathbf{C}$. The **analytic form of the Riemann Mapping Theorem 2.2.12** implies that Ω is biholomorphic to the unit disk. A biholomorphism is a homeomorphism, and a homeomorphism induces an isomorphism of fundamental groups. Since $\pi_1(D(0,1)) = \{[e]\}$ where e is the identity map by **Example 2.6.3**, Ω is simply connected. $\square$

Lemma 2.6.12. *Topological simple connectivity implies holomorphic simple connectivity.*

But wait! This is **Part I** of the **Riemann Mapping Theorem 2.2.2**. If U is connected and topologically simply connected, then U is homeomorphic to $D(0,1)$. The result is exactly what we need to show.

Proof of the Riemann Mapping Theorem, Part II. Let $\Omega \subseteq \mathbf{C}$ be topologically simply connected. Let $f \in H(\Omega)$. We need to show that f has a holomorphic antiderivative; i.e., there exists $F \in H(\Omega)$ with $F' = f$ on Ω.

Let $P \in \Omega$. If $Q \in \Omega$ and γ_1 and γ_2 are piecewise C^1 curves from P to Q, then the curve $\Gamma = {\gamma_2}^{-1} \cdot \gamma_1$ is a closed, piecewise C^1 curve in Ω. By the **Cauchy Integral Theorem for Non-Simply Connected, Connected, Open Sets 2.6.8** and the fact that Γ is homologous to 0, we get

$$0 = \oint_\Gamma f(z)dz = \oint_{\gamma_1} f(z)dz - \oint_{\gamma_2} f(z)dz.$$

Thus, we can define

$$F(Q) = \oint_\gamma f(z)dz,$$

where γ is any path from P to Q. It follows, as in the proof of **Morera's Theorem 1.9.24**, that $F \in H(\Omega)$ and $F' = f$ on Ω. $\square$

We have now proved the **Riemann Mapping Theorem 2.2.2**. If $\Omega \subseteq \mathbf{C}$ is a simply connected subset of $\mathbf{C}$, then $\Omega = \mathbf{C}$, or Ω is conformally equivalent to $D(0,1)$.

In fact, we have shown the following are equivalent, even though they may seem to put stronger and stronger restrictions on a set:

1. Ω is homeomorphic to $D(0,1)$,
2. Ω is simply connected,
3. Ω is holomorphically simply connected, and
4. Ω is conformally equivalent to $D(0,1)$.

Finally, we explore the notions of simple connectivity and the connectedness of the complement. The background idea will be to let $\gamma \subseteq \mathbf{C}$ be a closed curve. Then the winding number/index $\mathrm{Ind}_\gamma(a)$ is a continuous function in $a \in \mathbf{C} \setminus \gamma$. In particular, if $\Omega \subseteq \mathbf{C}$ is connected and $\gamma : [0,1] \to \Omega$ is closed and continuous, then $\mathrm{Ind}_\gamma(a)$ is constant on each connected component of $\mathbf{C} \setminus \gamma$. In particular in particular, we show that if C_1 is an unbounded component of $\mathbf{C} \setminus \Omega$, then $\mathrm{Ind}_\gamma(a) = 0$ for all $a \in C_1$. If Ω is bounded, then $\mathbf{C} \setminus \Omega$ has only one unbounded component. Therefore, if Ω is bounded and $\mathbf{C} \setminus \Omega$ has only one component, then that component is unbounded, and $\mathrm{Ind}_\gamma(a) \equiv 0$ for all $a \in \mathbf{C} \setminus \Omega$. We saw in **Lemma 2.6.7** that if Ω is simply connected, then $\mathrm{Ind}_\gamma(a) = 0$ for all $a \in \mathbf{C} \setminus \Omega$. These ideas are connected.

Lemma 2.6.13. *Let $\Omega \subseteq \mathbf{C}$ be connected and open, and let $\gamma : [0,1] \to \Omega$ be a continuous, closed curve. If C_1 is an unbounded component of $\mathbf{C} \setminus \Omega$, then $\mathrm{Ind}_\gamma(a) = 0$ for all $a \in C_1$.*

Proof. Recall that the winding number of γ around a is

$$\mathrm{Ind}_\gamma(a) = \frac{1}{2\pi i}\oint_\gamma \frac{1}{z-a}dz.$$

We know that the winding number is integer valued away from γ. C_1 lies in the complement of Ω and γ is in Ω, an open set, so $\mathrm{Ind}_\gamma(a) \in \mathbf{Z}$. We also know that the winding number is continuous away from γ, as $\frac{1}{z-a}$ is holomorphic away from γ. A continuous, integer valued function is constant, so $\mathrm{Ind}_\gamma(a)$ is constant on C_1. We must show it is zero somewhere; that will complete the proof.

Indeed, as C_1 is unbounded, send $|a| \to \infty$ keeping a in C_1. The limit passes through the integral as $\gamma \subseteq \Omega$ is far from C_1, and we have

$$\mathrm{Ind}_\gamma(a) = \lim_{|a|\to\infty} \mathrm{Ind}_\gamma(a) = \frac{1}{2\pi i}\oint_\gamma \lim_{|a|\to\infty}\frac{1}{z-a}dz = \frac{1}{2\pi i}\oint_\gamma 0dz = 0,$$

as desired. □

Theorem 2.6.14. *Let $\Omega \subseteq \mathbf{C}$ be a bounded, connected, open subset of $\mathbf{C}$. Then the following properties of Ω are equivalent:*

1. *Ω is simply connected,*

2. *$\mathbf{C} \setminus \Omega$ is connected, and*

3. *for each closed curve γ in Ω and $a \in \mathbf{C} \setminus \Omega$, $\mathrm{Ind}_\gamma(a) = 0$, so γ is homologous to 0.*

Proof. **Lemma 2.6.7** establishes that 1. implies 3. Additionally, by earlier discussion and **Lemma 2.6.13**, we have that 2. implies 3.

We now show that 3. implies 1. By the **Cauchy Integral Theorem for Non-Simply Connected, Connected, Open Sets 2.6.8** and **Proposition 1.16.21**, we are done.

We have left to show that 3. implies 2. This is the hardest part. We will show the following contrapositive: if $\Omega \subseteq \mathbf{C}$ is a bounded, connected, open set and if $\mathbf{C} \setminus \Omega$ is not connected, then there is a closed curve γ in Ω such that $\mathrm{Ind}_\gamma(a) \neq 0$ for some $a \in \mathbf{C} \setminus \Omega$.

To see this claim, since $\mathbf{C} \setminus \Omega$ is not connected, there exists a separation C_1 and C_2 of $\mathbf{C} \setminus \Omega$; i.e., $\mathbf{C} \setminus \Omega = C_1 \cup C_2$, where $C_1, C_2 \neq \emptyset$, $C_1 \cap C_2 = \emptyset$, and C_1 and C_2 are relatively closed in $\mathbf{C} \setminus \Omega$. However, since $\mathbf{C} \setminus \Omega$ is itself closed, C_1 and C_2 are closed in $\mathbf{C}$ as well. Only one of these sets can be unbounded, since

$$\left\{ z \mid |z| \geq 1 + \sup_{w\in\Omega} |w| \right\}$$

is connected. Let C_2 be the unbounded set, so C_1 is bounded.

Let $a \in C_1$, and set $d = \inf\{|z-w| \mid z \in C_1, w \in C_2\}$. Since C_1 is closed and bounded, hence compact, $d > 0$. To see this, set $\alpha(w) = \inf\{|z-w| \mid z \in C_1\}$. We claim $\alpha(w) > 0$. If not, there exists $(z_n) \subseteq C_2$ with $|z_n - w| \to 0$ as $n \to \infty$. But this means w is a limit point of C_1. Yet C_1 is closed, so it contains its limit points. If, then $w \in C_2$, then $w \in C_1 \cap C_2$, but $C_1 \cap C_2 = \emptyset$. Therefore indeed, $\alpha(w) > 0$ for $w \in C_2$.

We next claim that α is continuous. By the triangle inequality,

$$\begin{aligned} \alpha(w) &\leq |w-w'| + \alpha(w') \text{ and} \\ \alpha(w') &\leq |w-w'| + \alpha(w). \end{aligned}$$

Thus, $|\alpha(w) - \alpha(w')| \leq |w-w'|$, and α is Lipschitz with Lipschitz constant 1.

C_1 is compact, so it suffices to minimize α over $C_2 \cap \overline{D(0,R)}$ for suitably large R. $C_2 \cap \overline{D(0,R)}$ is compact, so α attains its minimum, hence, $d > 0$.

Next, cover the plane by a grid of closed squares of length $\frac{d}{10}$, with sides parallel to the axes. Without loss of generality, we may assume that a is at the center of one of these squares. Let S be the set of squares that intersect C_1 nontrivially. No square in S can intersect C_2 also, since if a $\frac{d}{10} \times \frac{d}{10}$ square intersected both C_1 and C_2, then $d \leq \frac{d}{10}\sqrt{2}$, a contradiction.

Let $S_1 \subseteq S$ be the set of squares that can be reached from the square containing a by a chain of squares having an edge in common. Orient the squares in S_1 counterclockwise, and consider all edges that belong to exactly one square in S_1. This set of edges is certainly the union of a finite number of piecewise linear, oriented, closed curves in Ω. Call them γ_j, $j \in \{1, ..., k\}$. The orientation of γ_j is determined by the orientation of the squares. Moreover, $\{\gamma_j\}$ are mutually disjoint, except perhaps at common vertices.

Now consider

$$\sum_{Q \in S_1} \frac{1}{2\pi i} \oint_{\partial Q} \frac{1}{\zeta - a} d\zeta.$$

Observe that the sum is 1, because if $a \in Q_a$, then

$$\frac{1}{2\pi i} \oint_{\partial Q_a} \frac{1}{\zeta - a} d\zeta = 1,$$

and if $Q \neq Q_a$, then

$$\frac{1}{2\pi i} \oint_{\partial Q} \frac{1}{\zeta - a} d\zeta = 0.$$

On the other hand, because a common edge of two squares in S_1 is counted once clockwise and once counterclockwise, it follows that

$$1 = \sum_{Q \in S_1} \frac{1}{2\pi i} \oint_{\partial Q} \frac{1}{\zeta - a} d\zeta = \sum_{j=1}^{k} \frac{1}{2\pi i} \oint_{\gamma_j} \frac{1}{\zeta - a} d\zeta.$$

Consequently, there is some closed curve in Ω around a with nonzero winding number, as we wished to show. $\square$

3 Post Semester Results

This section contains concepts that we tacked on to the end of the semester to explore interesting results from one variable complex analysis and to fill time. It is unlikely that any of the following will appear on a qualifying exam, but these results are still worthwhile to see.

3.1 The Prime Number Theorem

Definitions: Riemann Zeta function
Main Idea: The Prime Number Theorem gives an idea of the density of primes in the integers. We prove it.

Let $\mathcal{P}$ denote the set of all primes. Let $\pi(n)$ denote the number of primes between 2 and n. We will show that

$$\pi(n) \sim \frac{n}{\log n}.$$

Note that Gauss scribbled this formula in the margins of a book of tables AT AGE 14.

Theorem 3.1.1 (**The Prime Number Theorem**). *The expression $\pi(n)$ is asymptotically equal to $\frac{n}{\log n}$, in the sense that*

$$\lim_{n\to\infty} \frac{\pi(n)}{\frac{n}{\log n}} = 1.$$

It is a result due to Euler that if $\operatorname{Re} s > 1$, then

$$\prod_{p\in\mathcal{P}} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s),$$

where $\zeta(s)$ is the Riemann Zeta function. This provides a formal connection between the prime numbers and analysis.

We begin with the Riemann Zeta function.

Definition 3.1.2. For $\operatorname{Re} z > 1$, define the **Riemann Zeta function** to be

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \sum_{n=1}^{\infty} e^{-z\log n}.$$

The series converges normally on $\{z \mid \operatorname{Re} z > 1\}$, since $|n^z| = n^{\operatorname{Re} z}$. Thus, $\zeta \in H(\{z \mid \operatorname{Re} z > 1\})$.

Lemma 3.1.3 (**Euler Product Formula**). *For $\operatorname{Re} z > 1$, the infinite product*

$$\prod_{p\in\mathcal{P}} \left(1 - \frac{1}{p^z}\right)$$

converges, and

$$\frac{1}{\zeta(z)} = \prod_{p\in\mathcal{P}} \left(1 - \frac{1}{p^z}\right).$$

Proof. Since

$$\sum_{n=1}^{\infty} \frac{1}{n^z}$$

converges absolutely for $\operatorname{Re} z > 1$, the infinite product converges by **Corollary 2.5.11**; i.e.,

$$\prod_{p\in\mathcal{P}}\left(1-\frac{1}{p^z}\right)$$

converges.

Next,

$$\begin{aligned}
\zeta(z) &= \frac{1}{1^z}+\frac{1}{2^z}+\frac{1}{3^z}+\frac{1}{4^z}+\cdots, \text{ so}\\
\left(1-\frac{1}{2^z}\right)\zeta(z) &= \frac{1}{1^z}+\frac{1}{3^z}+\frac{1}{5^z}+\frac{1}{7^z}+\cdots, \text{ so}\\
\left(1-\frac{1}{3^z}\right)\left(1-\frac{1}{2^z}\right)\zeta(z) &= \frac{1}{1^z}+\frac{1}{5^z}+\frac{1}{7^z}+\frac{1}{11^z}+\cdots.
\end{aligned}$$

In fact, this method is the Sieve of Eratosthenes and shows that

$$\left(1-\frac{1}{{P_N}^z}\right)\left(1-\frac{1}{{P_{N-1}}^z}\right)\cdots\left(1-\frac{1}{2^z}\right)\zeta(z) = 1+\frac{1}{{P_{N+1}}^z}+\cdots.$$

Now, fix z with $\operatorname{Re} z > 1$. Let $\varepsilon > 0$ and choose N large enough so that

$$\sum_{n=N+1}^{\infty}\left|\frac{1}{n^z}\right| < \varepsilon.$$

Then, if $n \geq N$,

$$\left|\left(\prod_{j=1}^{n}\left(1-\frac{1}{{p_j}^z}\right)\right)\zeta(z)-1\right| < \varepsilon.$$

The result follows. $\square$

Lemma 3.1.4.

$$\sum_{p\in\mathcal{P}}\frac{1}{p}=\infty.$$

Proof. Observe that if $s \in (1,\infty)$, then $0 < \zeta(s) < \infty$. Also, by **Lemma 3.1.3**,

$$\frac{1}{\zeta(s)}=\prod_{p\in\mathcal{P}}\left(1-\frac{1}{p^s}\right).$$

The right hand side of above is decreasing as $s \to 1^+$.

Suppose to the contrary that

$$\sum_{p\in\mathcal{P}}\frac{1}{p}<\infty.$$

Then

$$\prod_{p\in\mathcal{P}}\left(1-\frac{1}{p}\right)$$

converges and is nonzero, by **Corollary 2.5.11**. Also, for all $s > 1$ and fixed $N \in \mathbf{N}$,

$$\prod_{\substack{p\in\mathcal{P}\\ p\leq N}}\left(1-\frac{1}{p^s}\right) \geq \prod_{\substack{p\in\mathcal{P}\\ p\leq N}}\left(1-\frac{1}{p}\right) \geq \prod_{p\in\mathcal{P}}\left(1-\frac{1}{p}\right).$$

Thus,

$$\lim_{s\to 1^+} \frac{1}{\zeta(s)} = \lim_{s\to 1^+} \prod_{p\in\mathcal{P}} \left(1 - \frac{1}{p^s}\right) \geq \prod_{p\in\mathcal{P}} \left(1 - \frac{1}{p}\right) > 0.$$

Hence, the monotone limit

$$\lim_{s\to 1^+} \zeta(s)$$

is finite. However, given $A > 0$, there exists $N \in \mathbf{N}$ so that

$$\sum_{n=1}^{N} \frac{1}{n} > A,$$

so

$$\lim_{s\to 1^+} \zeta(s) \geq \sum_{n=1}^{N} \frac{1}{n} > A.$$

Hence,

$$\lim_{s\to 1^+} \zeta(s)$$

cannot be finite, and we have reached a contradiction. $\square$

Note that the function $\pi(x)$ determines which numbers are prime, in the sense that $\pi(x)$ jumps by 1 whenever $x = p$ is prime. But a sum

$$\sum_{\substack{p\in\mathcal{P}\\ p\leq x}} f(p)$$

is determined by $\pi(x)$ as a Stieltjes integral. Namely,

$$\sum_{\substack{p\in\mathcal{P}\\ p\leq x}} f(p) = \int_0^x f(t)d\pi(t),$$

provided, for example, $f \in C([0,\infty))$.

Set

$$\vartheta(x) = \sum_{\substack{p\in\mathcal{P}\\ p\leq x}} \log p.$$

Lemma 3.1.5. *$\vartheta(x) \sim x$ if and only if $\pi(x) \sim \frac{x}{\log x}$.*

Thus, to prove the **Prime Number Theorem 3.1.1**, it will suffice to show that $\vartheta(x) \sim x$.

Proof. First, suppose that $\vartheta(x) \sim x$. Then

$$\vartheta(x) = \sum_{\substack{p\in\mathcal{P}\\ p\leq x}} \log p \leq \sum_{\substack{p\in\mathcal{P}\\ p\leq x}} \log x = \log x \pi(x).$$

Also, for $\varepsilon > 0$, we have

$$\vartheta(x) \geq \sum_{\substack{p\in\mathcal{P}\\ x^{1-\varepsilon}\leq p\leq x}} \log p \geq \sum_{\substack{p\in\mathcal{P}\\ x^{1-\varepsilon}\leq p\leq x}} (1-\varepsilon)\log x \geq (1-\varepsilon)\log x \left(\pi(x) - \pi\left(x^{1-\varepsilon}\right)\right).$$

Hence, for large x,

$$\frac{1}{x}\pi(x)\log x \geq \frac{\vartheta(x)}{x} \geq (1-\varepsilon)\frac{\log x}{x}\pi(x) - \pi\left(x^{1-\varepsilon}\right)\frac{\log x}{x}(1-\varepsilon).$$

But $\pi\left(x^{1-\varepsilon}\right) \leq x^{1-\varepsilon}$, so

$$\lim_{x\to\infty}\frac{\pi\left(x^{1-\varepsilon}\right)\log x}{x} = 0.$$

Thus, $\pi(x) \sim \frac{x}{\log x}$. The converse follows by reversing steps. □

Lemma 3.1.6. *If*

$$\lim_{x\to\infty}\int_1^x \frac{\vartheta(t)-t}{t^2}dt$$

exists, then $\vartheta(x) \sim x$.

Thus, to prove the **Prime Number Theorem 3.1.1**, it will suffice to show that

$$\lim_{x\to\infty}\int_1^x \frac{\vartheta(t)-t}{t^2}dt$$

exists.

Proof. Suppose there exists $\lambda > 1$ and a sequence $(x_j) \subseteq \mathbf{R}$ with $x_j \to \infty$ and $\vartheta(x_j) \geq \lambda x_j$. Since $\vartheta(x)$ is nondecreasing, setting $x = x_j$, we have

$$\int_x^{\lambda_x}\frac{\vartheta(t)-t}{t^2}dt \geq \int_x^{\lambda_x}\frac{\lambda x - t}{t^2}dt = \int_1^{\lambda}\frac{\lambda - t}{t^2}dt,$$

via a change of variables.

This estimate violates the convergence hypothesis, since

$$\lim_{j\to\infty}\int_{x_j}^{x_{j+1}}\frac{\vartheta(t)-t}{t^2}dt = 0.$$

Suppose now that there exists $0 < \lambda < 1$ and a sequence $(x_j) \subseteq \mathbf{R}$ with $x_j \to \infty$ and $\vartheta(x_j) \leq \lambda x_j$. Then setting $x = x_j$, we have

$$\int_{\lambda_x}^{x}\frac{\vartheta(t)-t}{t^2}dt \leq \int_{\lambda_x}^{x}\frac{\lambda x - t}{t^2}dt = \int_{\lambda}^{1}\frac{\lambda - t}{t^2}dt < 0,$$

again a contradiction. □

Lemma 3.1.7. *For some* $C > 0$ *and all* $x \geq 1$, $|\vartheta(x)| \leq C|x|$; *i.e.,* $\vartheta(x) = O(x)$.

Proof. For $N \in \mathbf{N}$,

$$(1+1)^{2N} = \sum_{m=0}^{2N}\binom{2N}{m} \geq \binom{2N}{N} \geq e^{\vartheta(2N)-\vartheta(N)},$$

since if $p \in \mathcal{P} \cap (N, 2N)$, p divides $(2N)!$ but p does not divide $N!$, and hence p divides $\binom{2N}{N} = \frac{(2N)!}{(N!)^2}$.

Taking logarithms yields that

$$\vartheta(2N) - \vartheta(N) \leq 2N\log 2,$$

and summing over $N = 2, 4, 8, ..., 2^k$ yields that

$$\vartheta\left(2^k\right) = 1 + \log 2\left(1 + \cdots + 2^k\right) \le 1 + \log 2\left(2^{k-1}\right) \le 3\log 2\left(2^k\right),$$

provided $k \ge 2$.

Thus, given $x \ge 2$ and $k \in \mathbf{N}$ so that $2^{k-1} \le x \le 2^k$, we conclude that

$$\vartheta(x) \le \vartheta\left(2^k\right) \le 3\log 2\left(2^k\right) \le (6\log 2)x.$$

□

We now begin to build some stronger connections to complex analysis.

Lemma 3.1.8. *The function* $\zeta(z) - \frac{1}{z-1}$ *defined initially for* $\operatorname{Re} z > 1$ *continues holomorphically to the set* $\{z \mid \operatorname{Re} z > 0\}$.

Proof. For $\operatorname{Re} z > 1$ fixed,

$$\zeta(z) - \frac{1}{z-1} = \sum_{n=1}^{\infty} \frac{1}{n^z} - \int_1^{\infty} \frac{1}{x^z} dx = \sum_{n=1}^{\infty} \int_n^{n+1} \left(\frac{1}{n^z} - \frac{1}{x^z}\right) dx.$$

By the Fundamental Theorem of Calculus,

$$\left|\int_n^{n+1} \left(\frac{1}{n^z} - \frac{1}{x^z}\right) dx\right| = \left|z \int_n^{n+1} \int_n^x \frac{1}{u^{z+1}} du dx\right| \le \max_{n \le u \le n+1} \left|\frac{z}{u^{z+1}}\right| = \frac{|z|}{n^{\operatorname{Re} z + 1}}.$$

Thus, the series

$$\sum_{n=1}^{\infty} \left(\int_n^{n+1} \left(\frac{1}{n^z} - \frac{1}{x^z}\right) dx\right)$$

converges absolutely for $\operatorname{Re} z > 0$ and normally on $\{z \mid \operatorname{Re} z > 0\}$, and thus can be used as a holomorphic extension of $\zeta(z) - \frac{1}{z-1}$ to $\{z \in \mathbf{C} \mid \operatorname{Re} z > 0\}$. □

Now, we have seen that

$$\vartheta(x) = \sum_{\substack{p \in \mathcal{P} \\ p \le x}} \log p$$

is of special interest to the **Prime Number Theorem 3.1.1**, but diverges as $x \to \infty$. Define, then, for $z \in \mathbf{C}$ and $\operatorname{Re} z > 1$, the function

$$\Phi(z) = \sum_{p \in \mathcal{P}} (\log p) p^{-z}.$$

Then, Φ converges absolutely and normally on $\{z \in \mathbf{C} \mid \operatorname{Re} z > 1\}$. Also note the following:

1. If p is large, then

$$\left|(\log p)p^{-z}\right| \le \left|p^{-z+\varepsilon}\right| \le \left|p^{-\operatorname{Re} z + \varepsilon}\right|;$$

   we need $\operatorname{Re} z - \varepsilon > 1$.

2. The series

$$\sum_{p \in \mathcal{P}} \left|p^{-s}\right|$$

   converges for all $s \in \mathbf{C}$ with $\operatorname{Re} s > 1$.

Then Φ is closely related to ζ. We proved in **Lemma 3.1.3** that

$$\zeta(z) = \prod_{p \in \mathcal{P}} \frac{1}{1 - p^{-z}}.$$

So, use logarithmic differentiation to see that

$$\frac{\zeta'(z)}{\zeta(z)} = -\sum_{p \in \mathcal{P}} \frac{\frac{\partial}{\partial z}\left[1 - p^{-z}\right]}{1 - p^{-z}} = -\sum_{p \in \mathcal{P}} \frac{p^{-z} \log p}{1 - p^{-z}} = -\sum_{p \in \mathcal{P}} \frac{\log p}{p^z - 1}.$$

Consequently,

$$\begin{aligned}
-\frac{\zeta'(z)}{\zeta(z)} &= \sum_{p \in \mathcal{P}} \frac{\log p}{p^z - 1} \\
&= \sum_{p \in \mathcal{P}} \frac{(\log p) p^z}{p^z \left(p^z - 1\right)} \\
&= \sum_{p \in \mathcal{P}} \frac{(\log p)\left(p^z - 1 + 1\right)}{p^z \left(p^z - 1\right)} \\
&= \sum_{p \in \mathcal{P}} \frac{\log p}{p^z \left(p^z - 1\right)} + \sum_{p \in \mathcal{P}} \frac{\log p}{p^z} \\
&= \sum_{p \in \mathcal{P}} \frac{\log p}{p^z \left(p^z - 1\right)} + \Phi(z).
\end{aligned}$$

The sum

$$\sum_{p \in \mathcal{P}} \frac{\log p}{p^z \left(p^z - 1\right)}$$

converges absolutely and normally on $\left\{z \mid \operatorname{Re} z > \frac{1}{2}\right\}$ by the same reasoning that showed that Φ is defined on $\{z \in \mathbf{C} \mid \operatorname{Re} z > 1\}$. Thus, since $\zeta(z) - \frac{1}{z-1}$ extends holomorphically to $\{z \in \mathbf{C} \mid \operatorname{Re} z > 0\}$, it follows that Φ extends meromorphically to $\left\{z \in \mathbf{C} \mid \operatorname{Re} z > \frac{1}{2}\right\}$. The extension has a pole at $z = 1$, since ζ has a simple pole at $z = 1$, and also has poles at the other zeros of ζ in $\left\{z \in \mathbf{C} \mid \operatorname{Re} z > \frac{1}{2}\right\}$.

Lemma 3.1.9. *The function $\zeta(z)$ is nonzero for $\{z \in \mathbf{C} \mid z = 1 + i\alpha, \alpha \in \mathbf{R}\}$. Thus, $\Phi(z) - \frac{1}{z-1}$ is holomorphic in a neighborhood of the line $\operatorname{Re} z = 1$.*

Proof. Since ζ has a pole at $z = 1$, and hence is nonzero, we only worry about $\alpha \neq 0$.

Suppose ζ has a zero of order μ at $1 + i\alpha$ and order ν at $1 + 2i\alpha$, using the convention that a zero of order 0 is a point where the function is nonzero. We will show that $\mu = 0$.

Using our formula for $-\frac{\zeta'(z)}{\zeta(z)}$, we recall that

$$-\frac{\zeta'(z)}{\zeta(z)} = \sum_{p \in \mathcal{P}} \frac{\log p}{p^z \left(p^z - 1\right)} + \Phi(z).$$

We show three facts:

1.

$$\lim_{\varepsilon \to 0^+} \varepsilon \Phi(1 + \varepsilon) = 1$$

holds, since $\Phi(z) - \frac{1}{z-1}$ is holomorphic near $z = 1$. Indeed, if $\zeta(z) = \frac{c_{-1}}{z-1} + h(z)$, then $\zeta'(z) = \frac{-c_{-1}}{(z-1)^2} + h'(z)$, and $(z-1)\frac{\zeta'(z)}{\zeta(z)}$ has a removable singularity at $z = 1$.

2.

$$\lim_{\varepsilon \to 0^+} \varepsilon \Phi(1 + \varepsilon \pm i\alpha) = -\mu$$

holds, since the final sum in the expression for $-\frac{\zeta'(z)}{\zeta(z)}$ converges, so its product with ε has limit 0 as $\varepsilon \to 0$.

Also note that at $1 - i\alpha$, ζ has a zero of order μ, since $\zeta(\overline{z}) = \overline{\zeta(z)}$.

3.

$$\lim_{\varepsilon \to 0^+} \varepsilon \Phi(1 + \varepsilon \pm 2i\alpha) = -\nu$$

holds, by the same argument as in 2.

Next, for $p \in \mathcal{P}$, $p^{\frac{i\alpha}{2}} + p^{\frac{-i\alpha}{2}}$ is real valued. Then

$$0 \leq \sum_{\substack{p \in \mathcal{P} \\ p > 2}} \frac{\log p}{p^{1+\varepsilon}} \left(p^{\frac{i\alpha}{2}} + p^{\frac{-i\alpha}{2}} \right)^4,$$

while

$$\sum_{p \in \mathcal{P}} \frac{\log p}{p^{1+\varepsilon}} \left(p^{\frac{i\alpha}{2}} + p^{\frac{-i\alpha}{2}} \right)^4 = \Phi(1+\varepsilon-2i\alpha) + 4\Phi(1+\varepsilon-i\alpha) + 6\Phi(1+\varepsilon) + 4\Phi(1+\varepsilon+i\alpha) + \Phi(1+\varepsilon+2i\alpha).$$

Multiplying by $\varepsilon > 0$ and sending $\varepsilon \to 0^+$ gives that

$$-\mu - 4\mu + 6 - 4\mu - \nu = 6 - 8\mu - 2\nu \geq 0.$$

This forces $\mu = 0$; i.e., $\zeta(1 + i\alpha) \neq 0$. The statement about Φ is already shown. $\square$

Lemma 3.1.10. *If* $\operatorname{Re} z > 1$, *then*

$$\Phi(z) = z \int_0^\infty e^{-zt} \vartheta\left(e^t\right) dt.$$

Proof. By the variable change $x = e^t$, $\frac{dx}{dt} = e^t$ (or $\frac{1}{x}\frac{dx}{dt} = 1$), we get

$$z \int_0^\infty e^{-zt} \vartheta\left(e^t\right) dt = z \int_1^\infty \frac{\vartheta(x)}{x^{1+z}} dx.$$

We will show that this latter expression equals $\Phi(z)$. As a reminder,

$$\vartheta(x) = \sum_{\substack{p \in \mathcal{P} \\ p \leq x}} \log p.$$

This means that it suffices to understand the contribution to the integral of

$$z \int_p^\infty \frac{\log p}{x^{z+1}} dx = \frac{\log p}{p^z},$$

with the above equality by an application of Fubini's Theorem. $\square$

Now, recall that in proving the **Prime Number Theorem 3.1.1**, we have reduced it to the case of proving that

$$I = \int_1^\infty \frac{\vartheta(x) - x}{x^2} dx$$

converges. Using the change of variables $x = e^t$, so $\frac{1}{x}\frac{dx}{dt} = 1$,

$$I = \int_0^\infty \frac{\vartheta\left(e^t\right) - e^t}{e^t} dt.$$

It therefore suffices to check the convergence of the integral

$$\int_1^\infty \left(\vartheta\left(e^t\right) e^{-t} - 1\right) dt.$$

Set $f(t) = \vartheta\left(e^t\right) e^{-t} - 1$. Then the formula

$$\Phi(z) = z \int_0^\infty e^{-zt} \vartheta\left(e^t\right) dt$$

would imply that the function

$$\int_0^\infty f(t) e^{-zt} dt$$

for $z \in \{w \in \mathbf{C} \mid \operatorname{Re} w > 0\}$ has an analytic continuation that is holomorphic on a neighborhood of $\{z \mid \operatorname{Re} z \geq 0\}$. Namely,

$$\begin{aligned}
\frac{1}{z+1}\Phi(z+1) &= \frac{1}{z+1}(z+1) \int_0^\infty e^{-(z+1)t} \vartheta\left(e^t\right) dt \\
&= \int_0^\infty e^{-zt} e^{-t} \vartheta\left(e^t\right) dt \\
&= \int_0^\infty e^{-zt} f(t) dt + \int_0^\infty e^{-zt} dt \\
&= \int_0^\infty e^{-zt} f(t) dt + \frac{1}{z}.
\end{aligned}$$

But we know that $\frac{1}{z+1}\Phi(z+1) - \frac{1}{z}$ is holomorphic on a neighborhood of $\{z \mid \operatorname{Re} z \geq 0\}$ by **Lemma 3.1.9**. So we have the following final result.

Lemma 3.1.11 (The Integral Theorem). *Let $f(t)$, $t \geq 0$, be a bounded, locally integrable function such that*

$$g(z) = \int_0^\infty f(t) e^{-zt} dt, \ \operatorname{Re} z > 0,$$

extends holomorphically to some neighborhood of $\{z \in \mathbf{C} \mid \operatorname{Re} z \geq 0\}$. Then,

$$\int_0^\infty f(t) dt$$

exists, and equals $g(0)$.

Before we prove this, note that with this lemma and $f(t) = \vartheta\left(e^t\right) e^{-t} - 1$, the proof of the **Prime Number Theorem 3.1.1** is complete. f is locally integrable (there are only jump discontinuities), and $\vartheta(t) \leq Ct$ for large t and some value C. Moreover, we have already checked that g has the requisite extension above.

Proof. Fix $T > 0$, and set

$$g_T(z) = \int_0^T f(t) e^{-zt} dt.$$

That $g \in H(\mathbf{C})$ is an easy consequence of **Morera's Theorem 1.9.24**. We must show that

$$\lim_{T\to\infty} g_T(0) = g(0).$$

Let $R > 0$ be large, $\delta > 0$ be small, and $C = \partial U$ where $U = \{z \in \mathbf{C} \mid |z| < R, \operatorname{Re} z > -\delta\}$. We choose δ small enough so that g is holomorphic in a neighborhood of $\overline{U}$.

By the **Cauchy Integral Formula 1.9.3**,

$$g(0) - g_T(0) = \frac{1}{2\pi i} \oint_C (g(z) - g_T(z))\, e^{zT} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z}.$$

On the semicircle $C_+ = C \cap \{z \mid \operatorname{Re} z > 0\}$, the integrand is bounded by $\frac{4B}{R^2}$, where $B = \max_{t\geq 0}|f(t)|$, since

$$|g(z) - g_T(z)| = \left|\int_T^\infty f(t)e^{-zt}dt\right| \leq B\int_T^\infty \left|e^{-zt}\right| dt = \frac{Be^{-\operatorname{Re} zT}}{\operatorname{Re} z}$$

for $\operatorname{Re} z > 0$, and

$$\begin{aligned}
\left|e^{zT}\left(1+\frac{z^2}{R^2}\right)\frac{1}{z}\right| &= e^{\operatorname{Re} zT}\left|\left(1+\frac{z^2}{R^2}\right)\frac{1}{z}\right| \\
&= e^{\operatorname{Re} zT}\left|\frac{R^2+z^2}{R^2}\cdot\frac{1}{z}\right| \\
&= \frac{e^{\operatorname{Re} zT}}{R}\left|\frac{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 + z^2}{R^2}\right| \\
&= \frac{e^{\operatorname{Re} zT}}{R}\left|\frac{2(\operatorname{Re} z)^2 + 2i(\operatorname{Re} z)(\operatorname{Im} z)}{R^2}\right| \\
&= \frac{e^{\operatorname{Re} zT}}{R^3}\left(4(\operatorname{Re} z)^4 + 4(\operatorname{Re} z)^2(\operatorname{Im} z)^2\right)^{\frac{1}{2}} \\
&= \frac{e^{\operatorname{Re} zT}}{R^2}2(\operatorname{Re} z).
\end{aligned}$$

Hence the contribution to $g(0) - g_T(0)$ from C_+ is bounded by $\frac{4\pi B}{R}$.

Let $C_- = C \cap \{z \mid \operatorname{Re} z < 0\}$. We analyze the g and g_T pieces separately. Since g_T is entire, we may replacethe path for this term with $C_-{}' = \{z \in \mathbf{C} \mid |z| = R, \operatorname{Re} z < 0\}$. The integral with g_T is then bounded in modulus by $\frac{4\pi B}{R}$ as before, since

$$|g_T(z)| \leq \left|\int_0^T f(t)e^{-zt}dt\right| \leq B\int_0^T \left|e^{-zt}\right| dt = \frac{Be^{-\operatorname{Re} zT}}{|\operatorname{Re} z|}$$

for $\operatorname{Re} z < 0$.

For g, we claim that the integral over C_- tends to 0 as $T \to \infty$, because the integrand is the product of $g(z)$ and $\left(1 + \frac{z^2}{R^2}\right)\frac{1}{z}$, which is independent of T, and the function $e^{zT} \to 0$ uniformly on compact sets as $T \to \infty$ in $\{z \in \mathbf{C} \mid \operatorname{Re} z < 0\}$. Hence,

$$\limsup_{T\to\infty}|g(0) - g_T(0)| \leq \frac{4\pi B}{R}.$$

As R was arbitrary, the theorem is proved. $\square$

3.2 The Phragman-Lindelöf Method

Definitions:

Main Idea: This section is an exploration of the consequences and failures of the Maximum Modulus Theorem. Like the Prime Number Theorem, this is supplementary.

The **Maximum Modulus Theorem 1.17.16** implies that if $\Omega \subseteq \mathbf{C}$ is a bounded region, then

$$\sup_{z\in\Omega}|f(z)| = \sup_{z\in\partial\Omega}|f(z)|$$

for all $f \in H(\Omega)$. A natural question to ask would be: what if Ω is *not* bounded? Then the answer is, unfortunately, it depends. We'll prove in this section a rather sharp requirement for a holomorphic function to achieve its maximum on the boundary or the interior of an unbounded open set. Here is one interesting example of the behavior on an unbounded set:

Example 3.2.1. Let $\Omega = \left\{z = x + iy \mid \frac{-\pi}{2} < y < \frac{\pi}{2}\right\}$. Then $\partial\Omega = \left\{z = x \pm i\frac{\pi}{2} \mid x \in \mathbf{R}\right\}$. Set $f(z) = e^{e^z}$. If $x \in \mathbf{R}$, then $f\left(x \pm i\frac{\pi}{2}\right) = e^{e^{x\pm i\frac{\pi}{2}}} = e^{e^x e^{\pm i\frac{\pi}{2}}} = e^{\pm ie^x}$. So $\left|f\left(x \pm i\frac{\pi}{2}\right)\right| = 1$, but $e^{e^x} \to \infty$ *very* fast as $x \to \infty$.

Note that we have seen this example before, in **Example 1.17.18**. We will see in **Theorem 3.2.5** that the *very* descriptor is important; any slower than e^{e^z} will fail.

Another example:

Example 3.2.2. If $f \in H(\mathbf{C})$ and $|f(z)| \le \left(1 + |z|^{\frac{1}{2}}\right)$, then $f \equiv C$. This follows from **Cauchy Estimates 1.11.1**.

Theorem 3.2.3. *Suppose $\Omega = \{x + iy \mid a < x < b\}$ so that $\overline{\Omega} = \{x + iy \mid a \le x \le b\}$. Let $f \in H(\Omega)$ and suppose that $|f(z)| \le B$ for all $z \in \Omega$. If*

$$M(x) = \sup_{y\in\mathbf{R}}\{|f(x + iy)|\},$$

$a \le x \le b$, then

$$M(x)^{b-a} \le M(a)^{b-x}M(b)^{x-a}$$

for all $a < x < b$.

Before the proof, note the following:

1. $M(x)^{b-a} \le M(a)^{b-x}M(b)^{x-a}$ means that $|f(z)| \le B$ can be replaced by $|f| \le \max\{M(a), M(b)\}$; i.e.,

$$\sup_{z\in\Omega}|f(z)| \le \sup_{z\in\partial\Omega}|f(z)|.$$

2. $M(x)^{b-a} \le M(a)^{b-x}M(b)^{x-a}$ can also be recast as follows:

   Corollary 3.2.4. *Under the hypotheses of **Theorem 3.2.3**, $\log M$ is a convex function on (a, b).*

Proof. So we prove **Theorem 3.2.3**.

We first assume that $M(a) = M(b1) = 1$. In this case, we must show that $|f(z)| \le 1$ for all $z \in \Omega$. For each $\varepsilon > 0$, define the auxiliary function $h_\varepsilon(z) = \frac{1}{1+\varepsilon(z-a)}$ for $z \in \overline{\Omega}$. Since $\mathrm{Re}(1+\varepsilon(z-a)) = 1+\varepsilon(x-a) \ge 1$ in $\overline{\Omega}$, $|h_\varepsilon(z)| = \frac{1}{|1+\varepsilon(z-a)|} \le \frac{1}{\mathrm{Re}(1+\varepsilon(x-a))} \le 1$ in $\overline{\Omega}$. So $|f(z)h_\varepsilon(z)| \le 1$ for $z \in \partial\Omega$.

Also, $|1 + \varepsilon(z - a)| \ge \varepsilon|y|$, so $|f(z)h_\varepsilon(z)| \le \frac{B}{\varepsilon|y|}$ for $z = x + iy \in \overline{\Omega}$.

Let R be the rectangle cut off from $\overline{\Omega}$ by the lines $y = \pm\frac{B}{\varepsilon}$; i.e.,

$$R = \left\{x + iy \in \mathbf{C} \mid a \le x \le b, \frac{-B}{\varepsilon} \le y \le \frac{B}{\varepsilon}\right\}.$$

It then follows from $|1 + \varepsilon(z - a)| \ge \varepsilon|y|$ and $|f(z)h_\varepsilon(z)| \le \frac{B}{\varepsilon|y|}$ that $|fh_\varepsilon| \le 1$ on ∂R; hence, on R as well, by the **Maximum Modulus Theorem 1.17.16**.

In addition, $|f(z)h_\varepsilon(z)| \le \frac{B}{\varepsilon|y|}$ also shows that $|f(z)h_\varepsilon(z)| \le 1$ on $\overline{\Omega} \setminus R$. Thus, $|f(z)h_\varepsilon(z)| \le 1$ for all $z \in \overline{\Omega}$ and all $\varepsilon > 0$. If we fix $z \in \Omega$ and send $\varepsilon \to 0$, then we see that $|f(z)| \le 1$, as desired.

We now turn to the general case. Set

$$g(z) = M(a)^{\frac{b-z}{b-a}} M(b)^{\frac{z-a}{b-a}},$$

so that for $M > 0$ and $w \in \mathbf{C}$, $M^w = e^{w \log M}$ and $\log M \in \mathbf{R}$. This means that $g \in H(\mathbf{C})$, g has no zeros, and $\frac{1}{g}$ is bounded in $\overline{\Omega}$. The latter is true because

$$\left|\frac{1}{g(z)}\right| = \left|\frac{1}{M(a)^{\frac{b-z}{b-a}} M(b)^{\frac{z-a}{b-a}}}\right|$$

and $\left|M(a)^{\frac{b-z}{b-a}}\right| = M(a)^{\mathrm{Re}\left(\frac{b-z}{b-a}\right)} = M(a)^{\frac{b-\mathrm{Re}(z)}{b-a}}$, and similarly for $\left|M(b)^{\frac{z-a}{b-a}}\right|$.

Moreover, $|g(a+iy)| = M(a)$ and $|g(b+iy)| = M(b)$. Finally, $\frac{f}{g}$ satisfies our previous assumptions. Thus, $\left|\frac{f}{g}\right| \leq 1$ in Ω, which implies that $M(x)^{b-a} \leq M(a)^{b-x} M(b)^{x-a}$, as desired. $\square$

Theorem 3.2.5. *Suppose $\Omega = \left\{x+iy \mid |y| < \frac{\pi}{2}\right\}$ so that $\overline{\Omega} = \left\{x+iy \mid |y| \leq \frac{\pi}{2}\right\}$. Suppose further that $f \in H(\Omega) \cap C(\overline{\Omega})$, and that there exist constants $\alpha < 1$ and $A < \infty$ so that*

$$|f(z)| \leq e^{Ae^{\alpha|x|}}$$

for $z = x+iy \in \Omega$, and

$$\left|f\left(x \pm i\frac{\pi}{2}\right)\right| \leq 1$$

for all $x \in \mathbf{R}$. Then $|f(z)| \leq 1$ for all $z \in \Omega$.

Note that we have already seen that **Theorem 3.2.5** fails with $\alpha = 1$ in **Example 3.2.1**, **Example 1.17.18**.

Proof. Choose $\beta > 0$ so that $\alpha < \beta < 1$. For $\varepsilon > 0$, set

$$h_\varepsilon(z) = \exp\left(-\varepsilon\left(e^{\beta z} + e^{-\beta z}\right)\right).$$

For $z \in \overline{\Omega}$,

$$\mathrm{Re}\left(e^{\beta z} + e^{-\beta z}\right) = \mathrm{Re}\left(e^{\beta x}e^{i\beta y} + e^{-\beta x}e^{-i\beta y}\right) = \left(e^{\beta x} + e^{-\beta x}\right)\cos(\beta y) \geq \left(e^{\beta x} + e^{-\beta x}\right)\delta,$$

where $\delta = \cos\left(\beta\frac{\pi}{2}\right) > 0$ since $\beta < 1$. Hence,

$$|h_\varepsilon(z)| \leq \exp\left(-\varepsilon\delta\left(e^{\beta x} + e^{-\beta x}\right)\right) < 1$$

for $z \in \overline{\Omega}$. It follows that $|fh_\varepsilon| \leq 1$ on $\partial\Omega$, and

$$|f(z)h_\varepsilon(z)| \leq \exp\left(Ae^{\alpha|x|} - \varepsilon\delta\left(e^{\beta x} + e^{-\beta x}\right)\right)$$

for $z \in \overline{\Omega}$.

Fix $\varepsilon > 0$. Since $\varepsilon\delta > 0$ and $\beta > \alpha$, the input of $\exp(\bullet)$ above tends to $-\infty$ as $x \to \pm\infty$. Hence, there exists x_0 such that $\exp\left(Ae^{\alpha|x|} - \varepsilon\delta\left(e^{\beta x} + e^{-\beta x}\right)\right) < 1$ if $|x| \geq x_0$. Since $|fh_\varepsilon| \leq 1$ on the boundary of the rectangle with vertices $\pm x_0$, $\pm i\frac{\pi}{2}$, the **Maximum Modulus Theorem 1.17.16** shows that $|fh_\varepsilon| \leq 1$ on this rectangle. Thus, $|fh_\varepsilon| \leq 1$ on $\overline{\Omega}$ for every $\varepsilon > 0$. As $\varepsilon \to 0$, $h_\varepsilon(z) \to 1$ for every $z \in \Omega$, and we may conclude that $|f(z)| \leq 1$ on Ω, as desired. $\square$

Theorem 3.2.6 (**Lindelöf**). *Suppose $\Gamma : [0,1] \to \overline{D(0,1)}$ is a curve such that $|\Gamma(t)| < 1$ if $t < 1$ and $\Gamma(1) = 1$. If $g \in H(D(0,1))$ and $g \in L^\infty(D(0,1))$ (i.e., g is bounded), often written $g \in H^\infty(D(0,1))$, and*

$$\lim_{t\to 1} g(\Gamma(t)) = L,$$

then g has a radial limit of L at 1; i.e.,

$$\lim_{\substack{t\in\mathbf{R} \\ t\to 1^-}} g(t) = L.$$

Proof. Without loss of generality, we may assume that $|g| < 1$, and $L = 0$. Let $\varepsilon > 0$ be given. There exists $t_0 < 1$ so that with $r_0 = \mathrm{Re}(\Gamma(t_0))$, we have $|g(\Gamma(t))| < \varepsilon$ and $\mathrm{Re}(\Gamma(t)) > r_0 > \frac{1}{2}$ when $t_0 < t < 1$.

Pick r such that $r_0 < r < 1$. Define h in $\Omega = D(0,1) \cap D(2r, 1)$ by

$$h(z) = g(z)\overline{g(\overline{z})}g(2r - z)\overline{g(2r - \overline{z})}.$$

Then $h \in H(\Omega)$, and $|h| < 1$, since $|g| < 1$. We claim that $|h(r)| < \varepsilon$. Since $h(r) = |g(r)|^4$, the theorem follows from such a claim.

To prove the claim, let $E_1 = \Gamma([t_1, 1])$, where $t_1 = \sup\{t > 0 \mid \mathrm{Re}(\Gamma(t)) = r\}$. Let E_2 be the reflection of E_1 across the real axis, and let E be $E_1 \cup E_2$ and the reflection of $E_1 \cup E_2$ across the line $x = r$. Now, note then that from $|g(\Gamma(t))| < \varepsilon$ and definition of $h(z)$, we have $|h(z)| < \varepsilon$ if $z \in \Omega \cap E$.

Pick $C > 0$, and define

$$h_C(z) = h(z)(1 - z)^C(2r - 1 - z)^C$$

for $z \in \Omega$. Put $h_C(1) = h_C(2r - 1) = 0$. If K is the union of E and the bounded components of the complement of E, then K is compact. h_C is continuous on K and holomorphic on the interior of K. Since $|h(z)| < \varepsilon$ on $z \in \Omega \cap E$, $|h_C| < \varepsilon$ on ∂K. Since the construction of E shows that $r \in K$, the **Maximum Modulus Theorem 1.17.16** implies that $|h_C(r)| < \varepsilon$. Sending $C \to 0$, we obtain $|h(r)| < \varepsilon$, as desired. ☐

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4.2 Appendix: Nice things to know, which may be fleshed out more in the future

During the two semesters, it was often called to attention specific theorems and proofs which are likely to appear on the qualifying exams. I did my best to mark them as they were mentioned. Additionally, for reassurance or in case I missed any, one can always look through past quals for common questions. Here, we list the results that I have marked, link to them in the notes, and provide a brief proof sketch of each.

Theorem (Morera's Theorem 1.9.24). *Suppose that $f : U \to \mathbf{C}$ is a continuous function on a connected, open set $U \subseteq \mathbf{C}$. Assume that for every closed, piecewise C^1 curve $\gamma : [0,1] \to U$, it holds that*

$$\oint_\gamma f(\zeta)d\zeta = 0.$$

Then $f \in H(U)$.

Proof sketch. Fix a point z_0 in U. Given another point w, let ψ be a piecewise C^1 curve from z_0 to w. Then

$$F(w) = \oint_\psi f(\zeta)d\zeta$$

is well-defined, because you can create a loop by running ψ and the inverse of any other piecewise C^1 curve from z_0 to w, and the hypothesis that f integrates to 0 along loops gives it to you.

Next, you have to show that F is C^1 and satisfies the Cauchy-Riemann equations. To see that $F \in C^1(U)$, let $w = x + iy$, take $F(w)$ as before, and move a little bit to the right of w via $w + h$, $h \in \mathbf{R}$. Let $\ell_h(t)$ be the curve connecting w and $w + h$ and let $\psi_h = \psi \cdot \ell_h$. Then

$$F(x+h, y) - F(x, y) = \oint_{\psi_h} f(\zeta)d\zeta - \oint_\psi f(\zeta)d\zeta = \oint_{\ell_h} f(\zeta)d\zeta = \int_0^h f(w+s)ds.$$

Then if $F = U + iV$, show that U is C^1 via the obvious difference quotient, which ends up being the average value integral

$$\frac{1}{h}\int_0^h \operatorname{Re} f(z+s)ds,$$

and since f is continuous, $\frac{\partial U}{\partial x}(z) = \operatorname{Re} f(z)$. You can do the same for V; indeed $\frac{\partial U}{\partial y}(z) = -\operatorname{Im} f(z)$, $\frac{\partial V}{\partial x} = \operatorname{Im} f(z)$, and $\frac{\partial V}{\partial y} = \operatorname{Re} f(z)$. Thus the Cauchy-Riemann equations are satisfied. And since f is continuous, so are the partials of U and V. So $F \in C^1$.

Therefore, $F \in H(U)$, so $F' = f$ is holomorphic as well, as desired! □

Theorem (Liouville's Theorem 1.11.4). *A bounded, entire function is constant.*

Proof sketch. If f is bounded by M and entire, then Cauchy Estimates say that at some point in $\mathbf{C}$, the first derivative is bounded in modulus by $\frac{M}{r}$ for an arbitrary $r > 0$. Send $r \to \infty$ to bound the derivative by 0 there. But your choice of point is arbitrary too, so the first derivative is 0 everywhere. As $\mathbf{C}$ is connected, $f \equiv C$. □

Theorem (The Fundamental Theorem of Algebra 1.11.6). *Let $p(z)$ be a nonconstant holomorphic polynomial. Then p has a root; i.e., there exists $\alpha \in \mathbf{C}$ such that $p(\alpha) = 0$.*

Proof sketch. By contradiction. If not, $\frac{1}{p}$ is entire. Since $|p(z)| \to \infty$ as $|z| \to \infty$, $\frac{1}{p} \to 0$, so $\frac{1}{p}$ is bounded, and by Liouville, constant, so p is too. Contradiction. □

Theorem (Rouché's Theorem 1.17.12). *Suppose that f and g are holomorphic on U and $U \subseteq \mathbf{C}$ is open. Suppose further that $\overline{D(z_0, r)} \subseteq U$, and for each $\zeta \in \partial D(z_0, r)$,*

$$|f(\zeta) + g(\zeta)| < |f(\zeta)| + |g(\zeta)|.$$

Then

$$\frac{1}{2\pi i}\oint_{\partial D(z_0,r)} \frac{f'(\zeta)}{f(\zeta)}d\zeta = \frac{1}{2\pi i}\oint_{\partial D(z_0,r)} \frac{g'(\zeta)}{g(\zeta)}d\zeta.$$

Proof sketch. The line between $f(\zeta)$ and $g(\zeta)$, $f_t(\zeta) = tf(\zeta)+(1-t)g(\zeta)$, is never zero for any $\zeta \in \partial D(z_0, r)$ by the inequality hypothesis. Define I_t the argument principle of f_t for every t. As $f_t(\zeta) \in H(U)$ is never zero, the integrand is bounded and continuous in t, so the argument principle is continuous in t and obviously integer valued, hence constant. Thus $I_0 = I_1$. $\square$

Theorem (**Hurwitz's Theorem 1.17.14**). *Let $U \subseteq \mathbf{C}$ be connected and open, and let $f_j : U \to \mathbf{C}$ be holomorphic and nonvanishing. If (f_j) converges uniformly on compact subsets of U to f_0, then either $f_0(z) \equiv 0$ for all $z \in U$, or $f_0(z) \neq 0$ for all $z \in U$.*

Proof sketch. By contradiction, assume that f_0 has a zero but isn't identically zero. Pick a suitable r where f_0 is nonzero on the punctured disk around the zero, and then half that r and look at some argument principles. The argument principle for f_0 is nonzero, but the argument principle for all f_j is zero. Since $f_j \to f_0$ normally, ${f_j}' \to {f_0}'$ and $\frac{1}{f_j} \to \frac{1}{f_0}$ normally. Normal (uniform) convergence lets us commute limits, so push through the integral to reach a contradiction. $\square$

Theorem (**Schwarz's Lemma 1.18.1**). *Let $f \in H(D(0,1))$. Assume that $|f(z)| \leq 1$ for all $z \in D(0,1)$ and that $f(0) = 0$. Then $|f(z)| \leq |z|$, and $|f'(0)| \leq 1$. If either $|f(z)| = |z|$ for some $z \neq 0$, or $|f'(0)| = 1$, then f is a rotation; i.e., $f(z) = \alpha z$ for some $\alpha \in \mathbf{C}$, $|\alpha| = 1$.*

Proof sketch. Let $g(z) = \frac{f(z)}{z}$. g is holomorphic away from 0 and as we approach 0, by construction we're $f'(0)$. So consider g to be holomorphic via Riemann Removable Singularities by making $g(0) = f'(0)$.

Consider $\overline{D(0, 1-\varepsilon)}$. On its boundary, we've hypothesized that $|f| \leq 1$, so $|g| \leq \frac{1}{1-\varepsilon}$. So Maximum Modulus says that g is bounded by $\frac{1}{1-\varepsilon}$ inside of the disk too. Send ε to 0 to see that $|g| \leq 1$ in $D(0,1)$; i.e., $|f(z)| \leq |z|$, and $1 \geq |g(0)| = |f'(0)|$.

For the second part of the theorem, first assume $|f(z)| = |z|$, $z \neq 0$. Then $|g(z)| = 1$ in $D(0,1)$ so Maximum Modulus says $g \equiv \alpha$ where $|\alpha| = 1$. So $f(z) = \alpha z$. Second assume $|f'(0)| = 1$, then $|g(0)| = 1$ and the same argument applies. $\square$

Note also that it is almost surely the case that we will need to evaluate an integral using the **Residue Theorem 1.16.24** and an appropriate choice of complex function and contour. It may be prudent, therefore, to make sure the following examples are clear:

- **Example 1.16.27**
- **Example 1.16.28**
- **Example 1.16.29**
- **Example 1.16.30**
- **Example 1.16.31**
- **Example 1.16.32**